Computational Algebra

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Preface

Computational algebra is concerned with finding algorithms for computing with the objects that play a role in algebra. The area is divided into sub-areas in the same way as algebra is divided into sub-areas. So we have, for example, computational number theory, computatonal commutative algebra, computational group theory, and so on. These areas are of course not completely separate (as are the sub-areas of algebra, and indeed, of the whole of science). For example, when working on algorithms for groups it may be necessary to look at representations of these groups, which depend on a field and thus may lead to problems in computational number theory.

These notes are divided into four chapters, each devoted to a particular type of algorithmic problem:

- Chapter 1 deals with algorithms for working with ideals in multivariate polynomial rings. The main object of interest is a Gröbner basis of an ideal. We discuss how to compute a Gröbner basis, and describe several applications.
- In Chapter 2 we look at the very classical problem of finding the prime factors of an integer. Also because of the important applications in cryptography, many algorithms have been developed for this task. Here we cannot go into all of them (that would require a rather voluminous book). We describe two modern algorithms: the first one based on continued fractions, the second based on the theory of elliptic curves.
- Chapter 3 focusses on the same problem, but now for univariate polynomials. This is very different compared to the integer factorization problem as here the base field plays a major role. We describe algorithms for polynomials over a finite field, and for polynomials over the field of rational numbers.
- Chapter 4 has an introduction to the celebrated LLL algorithm, for finding a reduced basis of a lattice. This algorithm has many surprising applications, from cryptography to the GPS system. We describe an application in cryptography.

I believe that these subjects are of sufficient general interest to be part of an introductory course on computational algebra. There are of course many areas of computational algebra that are not covered in these notes.

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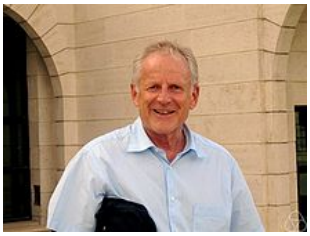
Chapter 1

Gröbner Bases

The term "Gröbner bases" was introduced by Bruno Buchberger



in his 1965 Ph.D. thesis (*An algorithm for finding the basis elements of the residue class ring of a zero dimensional polynomial ideal*, University of Innsbruck) under the supervision of Wolfgang Gröbner. However, similar notions in different settings were around before that date, and have been developed since. In 1900 Gordan appears to have used a very similar concept. Furthermore, a "Gröbner basis theory" for free Lie algebras was developed by Shirshov in 1962 (Some algorithmic problems for Lie algebras, *Sib. Mat. Zh*). Also, the so-called Knuth-Bendix algorithm (Knuth and Bendix, Simple word problems in universal algebra, 1970) is a similar development in the theory of finitely-presented groups.

In this chapter we describe the theory of Gröbner bases, and some of its applications, for ideals in a polynomial ring.

Throughout we use the notation $\mathbb{N} = \mathbb{Z}_{\geq 0}$.

1.1 Gröbner bases

1.1.1 Polynomial rings

Throughout k will denote a field, and $k[x_1, \ldots, x_n]$ will be the polynomial ring in n indeterminates, with coefficients in k. That is

$$k[x_1, \ldots, x_n] = \{ \sum_{i_1, \ldots, i_n} c_{i_1 \ldots i_n} x_1^{i_1} \cdots x_n^{i_n} \mid c_{i_1 \ldots i_n} \in k \}.$$

Polynomials are added and multiplied in the obvious way making $k[x_1, \ldots, x_n]$ into a commutative ring with unity.

Example 1.1.1 In the cases where we have one, two or three indeterminates we also write, respectively, $k[x]$, $k[x,y]$ and $k[x,y,z]$.

A more compact notation for the exponents is useful on many occasions. For $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \in \mathbb{N}$, we define $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Such an element is called a *monomial*. So in general we can write a polynomial as $\sum_\alpha c_\alpha x^\alpha$.

Definition 1.1.2 *Let R be a commutative ring. A subset $I \subset R$ is called an* ideal *if*

- $0 \in I$;
- *if $f, g \in I$ then also $f + g \in I$;*
- *if $f \in I$ and $g \in R$ then $gf \in I$.*

Let $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$ and set

$$I = \{\sum_{i=1}^{s} g_i f_i \mid g_i \in k[x_1, \ldots, x_n]\}.$$

Then obviously I is an ideal of $k[x_1, \ldots, x_n]$ (in fact, it is the smallest ideal of $k[x_1, \ldots, x_n]$ containing $f_1, \ldots, f_s$). Later we will see that every ideal of $k[x_1, \ldots, x_n]$ is of this form.

Notation. We will denote the ideal I above with $I = \langle f_1, \ldots, f_s \rangle$, and say that I is generated by $f_1, \ldots, f_s$.

Ideal Membership Problem: Given $I = \langle f_1, \ldots, f_s \rangle$ and $g \in k[x_1, \ldots, x_n]$ decide whether $g \in I$.

Example 1.1.3 Let $R = k[x,y,z]$, $f_1 = xyz - xy$, $f_2 = x^2y - yz$ and $g = yz^2 - yz$. The question is whether $g \in \langle f_1, f_2 \rangle$.

Observe that

$$xf_1 - zf_2 = -x^2y + yz^2$$

so we add f_2 and find

$$xf_1 - zf_2 + f_2 = yz^2 - yz = g$$

and hence $g = xf_1 + (1 - z)f_2 \in \langle f_1, f_2 \rangle$.

Example 1.1.4 Let $I \subset k[x,y,z]$ be the ideal generated by $f_1 = x^2yz - yz - x$, $f_2 = xy^2z - xy - y$, $f_3 = xyz^2 - xy - z$. Then $x^2 - z^2$ and $y - z$ lie in I. But it certainly is not obvious.

We see that checking whether $g \in \langle f_1, \ldots, f_s \rangle$ can be a rather complicated thing. But at least we know that we have to find h_i with $g = \sum_i h_i f_i$. To show that $g \notin \langle f_1, \ldots, f_s \rangle$ seems even more difficult.

1.1.2 Polynomials in one indeterminate

If we are dealing with polynomials in a single indeterminate, then the fundamental problem has an easy solution.

Lemma 1.1.5 *Let $I \subset k[x]$ be an ideal. Then there exists $h \in k[x]$ with $I = \langle h \rangle$.*

PROOF. Let h be a nonzero element of I, of minimal degree. Let $f \in I$. Then $f = qh + r$ with $q, r \in k[x]$ and $\deg(r) < \deg(h)$.

Since $r = f - qh \in I$ we get that $r = 0$ as h is of minimal degree in I. It follows that $f = qh$. □

Now let $I = \langle h \rangle \subset k[x]$ and $f \in k[x]$. From the proof of the preceding lemma we get the following procedure to check whether $f \in I$:

1) Write $f = qh + r$ with $\deg(r) < \deg(h)$. (Polynomial division.)

2) If $r = 0$ then $f \in I$ otherwise $f \notin I$.

(Note that if I is generated by h, then h is of minimal degree in I.)

We want to find some analogous procedure in the multivariate case. The key remark is that in the case of one indeterminate we use the degree of a polynomial. In other words, we use an ordering on the monomials: $x^m < x^n$ if $m < n$. We start by generalising that to the multivariate situation.

1.1.3 Monomial orders

An *order* on a set A is a relation $\leq$ such that for all $a, b, c \in A$ we have

$$\begin{aligned} &a \leq a \\ &a \leq b,\ b \leq a \Rightarrow a = b \\ &a \leq b,\ b \leq c \Rightarrow a \leq c. \end{aligned}$$

The order $\leq$ is said to be *total* if for all $a, b \in A$ we have $a \leq b$ or $b \leq a$. An order that is not necessarily total is called *partial*.

Recall the notation $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$.

Definition 1.1.6 *An order $\leq$ on the set $\{x^\alpha\}$ of the monomials of $k[x_1, \ldots, x_n]$ is called a* monomial order *if*

1) *$\leq$ is <u>total</u>, that is, for all α, β we have $x^\alpha \leq x^\beta$ or $x^\beta \leq x^\alpha$.*

2) *$\leq$ is <u>multiplicative</u>, that is, if $x^\alpha \leq x^\beta$ then $x^\alpha x^\gamma \leq x^\beta x^\gamma$ for all γ.*

3) *There are no infinite descending chains $x^{\alpha(1)} > x^{\alpha(2)} > \ldots$ (in other words, $\leq$ is a* well-ordering*).*

We identify $\{x^\alpha \mid \alpha \in \mathbb{N}^n\}$ and $\mathbb{N}^n$ by $x^\alpha \leftrightarrow \alpha$.

So from an order $\leq$ on the set of monomials we get an order on $\mathbb{N}^n$ by $\alpha < \beta$ if $x^\alpha < x^\beta$. Then the conditions for $\leq$ to be a monomial order translate to:

1) $\leq$ is <u>total</u>.

2) If $\alpha \leq \beta$ then $\alpha + \gamma \leq \beta + \gamma$ for all γ.

3) There are no infinite descending chains.

In the following, when we speak about monomial orders we freely interchange between an order on the set of monomials and on $\mathbb{N}^n$, whichever is the most convenient.

The following observation is useful.

Lemma 1.1.7 *Set $\gamma_0 = (0, \ldots, 0) \in \mathbb{N}^n$ and let $\leq$ be a monomial order. Then $\alpha > \gamma_0$ for all $\alpha \in \mathbb{N}^n$, $\alpha \neq \gamma_0$.*

Proof. If $\alpha < \gamma_0$ then $\alpha + \alpha < \alpha + \gamma_0 = \alpha$. In the same way we find that $(k+1)\alpha < k\alpha$. So we have constructed an infinite descending chain. □

Example 1.1.8 (Lexicographical order.) This order is denoted $<_{\text{lex}}$. By definition, $\alpha <_{\text{lex}} \beta$ if $\alpha_i < \beta_i$ where i is the minimal index with $\alpha_i \neq \beta_i$.

For example in $k[x, y, z]$ we have

$$xy^2z^{100} <_{\text{lex}} x^2yz$$

and

$$xyz <_{\text{lex}} xy^3.$$

The claim is that $<_{\text{lex}}$ is a monomial order.

Proof. 1) It is obvious that it is a total order: if $\alpha \neq \beta$ then there is an index i with $\alpha_i \neq \beta_i$.

2) Suppose that $\alpha <_{\text{lex}} \beta$ and let i be minimal with $\alpha_i \neq \beta_i$, and hence $\alpha_i < \beta_i$. Then i is also the minimal index with $(\alpha + \gamma)_i \neq (\beta + \gamma)_i$ and

$$(\alpha + \gamma)_i = \alpha_i + \gamma_i < \beta_i + \gamma_i = (\beta + \gamma)_i.$$

Therefore $(\alpha + \gamma) <_{\text{lex}} (\beta + \gamma)$.

3) The most difficult thing to show is that there are no infinite descending chains. For this we use induction on n (i.e., the number of indeterminates).

For $n = 1$ it is obvious that there are no infinite descending chains (here the order reduces to the normal degree order).

Assume that there are no infinite descending chains with respect to $<_{\text{lex}}$ when there are $n - 1$ indeterminates. Also suppose that there is an infinite descending chain $\alpha(1) >_{\text{lex}} \alpha(2) >_{\text{lex}} \ldots$, with $\alpha(i) \in \mathbb{N}^n$.

Observe that $0 \leq \alpha(i)_1 \leq \alpha(1)_1$.

Hence

$$\{\alpha(i) \mid i \geq 1\} = \bigcup_{k=0}^{\alpha(1)_1} C_k$$

where $C_k = \{\alpha(i) \mid i \geq 1,\ \alpha(i)_1 = k\}$.

So there has to be a k_0 with C_{k_0} infinite. Then we set $\widetilde{C}_{k_0} = \{(\alpha_2, \ldots, \alpha_n) \mid \alpha \in C_{k_0}\}$.

But this is an infinite descending chain with respect to the $<_{\text{lex}}$ order on $n-1$ indeterminates. So we have our contradiction. □

Example 1.1.9 (Graded lexicographical order.) Notation $<_{\text{glex}}$.

Define $|\alpha| = \alpha_1 + \ldots + \alpha_n$; this is the degree of the monomial x^α. By definition $\alpha <_{\text{glex}} \beta$ if

- $|\alpha| < |\beta|$

- or $|\alpha| = |\beta|$ and $\alpha <_{\text{lex}} \beta$.

It is obvious that $<_{\text{glex}}$ is a monomial order.

1.1.4 Polynomial division for multivariate polynomials

As before we write $R = k[x_1, \ldots, x_n]$. In this section we fix a monomial order $\leq$ on $\mathbb{N}^n$.

Let $f = \sum_\alpha c_\alpha x^\alpha \in k[x_1, \ldots, x_n]$, and let α be maximal with respect to $\leq$ with $c_\alpha \neq 0$. Then x^α is called the *leading monomial* of f, and denoted $\mathrm{LM}(f)$, whereas c_α is called the *leading coefficient* and denoted by $\mathrm{LC}(f)$.

With this notation we have the following division algorithm, which is a generalisation of the division algorithm for polynomials in one indeterminate.

Algorithm 1.1.10 (Polynomial division with remainder)
Given: $f_1, \ldots, f_s \in R$, each $f_i \neq 0$, $g \in R$.
We compute: $r \in R$ such that $g = h_1 f_1 + \ldots + h_s f_s + r$ with $h_i \in R$ and no monomial of r is divisible by an $\mathrm{LM}(f_i)$.

1) *Set $\bar{g} := g$, $r := 0$.*

2) *If $\bar{g} = 0$ we stop.*

3) *Write $x^\gamma = \mathrm{LM}(\bar{g})$, $c = \mathrm{LC}(\bar{g})$. If there is an f_i such that $\mathrm{LM}(f_i)$ divides x^γ then we set*

$$\bar{g} := \bar{g} - \frac{c}{\mathrm{LC}(f_i)} x^\beta f_i$$

where $x^\beta \mathrm{LM}(f_i) = x^\gamma$. Otherwise we set

$$\bar{g} := \bar{g} - cx^\gamma \quad \text{and} \quad r := r + cx^\gamma.$$

Return to 2).

Proposition 1.1.11 *The algorithm 1.1.10 terminates correctly.*

Proof. In every round of the algorithm $\mathrm{LM}(\bar{g})$ decreases. Hence the algorithm terminates as there are no infinite descending chains.

Note that throughout the execution of the algorithm we have $g - (\bar{g} + r) \in I$, where $I = \langle f_1, \ldots, f_s \rangle$. Indeed, this is surely true initially (when $\bar{g} = g$, $r = 0$). Assume it holds for $\bar{g}$ and r. If they get replaced by $\bar{g}_1$, r_1, with

$$\bar{g}_1 = \bar{g} - \frac{c}{\mathrm{LC}(f_i)} x^\beta f_i \quad \text{and} \quad r_1 = r,$$

then

$$g - (\bar{g}_1 + r_1) = g - (\bar{g} + r) + \frac{c}{\mathrm{LC}(f_i)} x^\beta f_i$$

so also $g - (\bar{g}_1 + r_1) \in I$. If on the other hand $\bar{g}_1 = \bar{g} - cx^\gamma$ and $r_1 = r + cx^\gamma$ then

$$g - (\bar{g}_1 + r_1) = g - (\bar{g} + r) \in I.$$

So the property continues to hold in the next round of the algorithm. (The property $g - (\bar{g} + r) \in I$ is said to be a *loop-invariant* of the algorithm.)

So upon termination we have $\bar{g} = 0$ and $g - r \in I$ and therefore

$$g - r = h_1 f_1 + \ldots + h_s f_s.$$

By construction we see that no $\mathrm{LM}(f_i)$ divides a monomial in r. □

Remark 1.1.12 We note that the algorithm, and hence its output, depends strongly on the chosen monomial order. In the next example we see that there is another element of choice in the algorithm that can also affect the output.

Example 1.1.13 Let $f_1 = xy + 1$, $f_2 = y + 1$ and $g = xy - y$. We use the monomial order $<=<_{\mathrm{glex}}$.

We can perform the algorithm in two ways:

- First alternative:

  1) $\bar{g} = xy - y$, $r = 0$;
  2) $\bar{g} = \bar{g} - f_1 = -y - 1$;
  3) $\bar{g} = \bar{g} + f_2 = 0$;
  4) $r = 0$.

- Second alternative:

  1) $\bar{g} = xy - y$, $r = 0$;
  2) $\bar{g} = \bar{g} - xf_2 = -x - y$;
  3) $\bar{g} = \bar{g} + x = -y$, $r = -x$;
  4) $\bar{g} = \bar{g} + f_2 = 1$;
  5) $\bar{g} = \bar{g} - 1 = 0$, $r = -x + 1$;
  6) $r = -x + 1$.

Remark 1.1.14 Let $G \subset k[x_1, \ldots, x_n]$ be a set of polynomials, and $f \in k[x_1, \ldots, x_n]$. Let $g \in G$ be nonzero and let x^α a monomial of f such that $\mathrm{LM}(g)$ divides x^α. Let x^β be such that $x^\alpha = x^\beta \mathrm{LM}(g)$, and set $\bar{f} = f - \frac{c}{\mathrm{LC}(g)} x^\beta g$, where c is the coefficient of x^α in f. Then we say that f *reduces* to $\bar{f}$ modulo G, and we write $f \xrightarrow{g} \bar{f}$. Division with remainder can also be described as a maximal sequence of reduction steps. The second alternative of the previous example becomes

$$g \xrightarrow{f_2} -x - y \xrightarrow{f_2} -x + 1.$$

Definition 1.1.15 *Let r be the output of Algorithm 1.1.10 with input $f_1, \ldots, f_s \in R$, and $g \in R$. Then r is called a* remainder *of g modulo $f_1, \ldots, f_s$.*

From this example we see that there can be more than one remainder of g modulo $f_1, \ldots, f_s$. So the main problem is not solved as easily as in the univariate case. The main idea is to find a different generating set $G = \{g_1, \ldots, g_t\}$ of the ideal $I = \langle f_1, \ldots, f_s \rangle$, with the property that the remainder of a $g \in R$ modulo G is unique, and moreover it is zero if and only if $g \in I$. Such a G will be called a Gröbner basis.

First we study what happens in a relatively easy case.

1.1.5 Monomial ideals

Definition 1.1.16 *Let $A \subseteq \mathbb{N}^n$ (possibly infinite). Then the ideal*

$$I = \langle x^\alpha \mid \alpha \in A \rangle \subset R = k[x_1, \ldots, x_n]$$

is called a monomial ideal.

Lemma 1.1.17 *Let $I = \langle x^\alpha \mid \alpha \in A \rangle$ be a monomial ideal. Then $h \in I$ if and only if all of its monomials lie in I.*

Proof. "$\Leftarrow$". Obvious.

"$\Rightarrow$". Let $h \in I$ and write $h = \sum_{\alpha \in A} h_\alpha x^\alpha$ with $h_\alpha \in R$, only finitely many of which are nonzero.

Let x^γ be a monomial of h. So x^γ is a monomial of at least one $h_\alpha x^\alpha$. But

$$h_\alpha = \sum_{i=1}^{m} \mu_i x^{\beta(i)} \quad \Longrightarrow \quad h_\alpha x^\alpha = \sum_{i=1}^{m} \mu_i x^{\beta(i)+\alpha}.$$

So there is an i with

$$x^\gamma = x^{\beta(i)} x^\alpha \in I.$$

□

Lemma 1.1.18 *Let the notation be as in the preceding lemma. We have $x^\gamma \in I$ if and only if $x^\gamma = x^\beta x^\alpha$ for certain $\beta \in \mathbb{N}^n$ and $\alpha \in A$.*

Proof. Analogous to the proof of Lemma 1.1.17. □

Now we define a partial order $\preceq$ on $\mathbb{N}^n$ by $\alpha \preceq \beta$ if $\alpha_i \leq \beta_i$ for all i.

Example 1.1.19 We have $(1,2,3) \preceq (1,3,5)$. Furthermore, $(1,2,3) \not\preceq (1,3,2)$ and $(1,3,2) \not\preceq (1,2,3)$.

Hence not all elements are comparable with the order $\preceq$. We say that $\alpha, \beta \in \mathbb{N}^n$ with $\alpha \not\preceq \beta$ and $\beta \not\preceq \alpha$ are *uncomparable* with respect to $\preceq$.

A set $A \subseteq \mathbb{N}^n$ of which any pair of elements is uncomparable is called an *antichain*.

Lemma 1.1.20 *Every antichain in $\mathbb{N}^n$ is finite.*

Proof. For $n = 1$ this is obvious as in that case $\preceq$ is equal to the normal order $\leq$ on the integers. (So an antichain has at most one element.)

Let $n > 1$ be minimal such that there is an infinite antichain $A \subseteq \mathbb{N}^n$. Let $\alpha \in A$. Then for all $\beta \in A$ with $\beta \neq \alpha$ there exists i with $\beta_i < \alpha_i$ (and j with $\beta_j > \alpha_j$), because α and β are not comparable. Then

$$A \setminus \{\alpha\} = \bigcup_{i=1}^{n} \{\beta \in A \mid \beta_i < \alpha_i\}.$$

So since A is infinite, at least one set $B_{i_0} = \{\beta \in A \mid \beta_{i_0} < \alpha_{i_0}\}$ is infinite. Write $\delta = \alpha_{i_0}$.

Then $B_{i_0} = \cup_{j=0}^{\delta-1}\{\beta \in A \mid \beta_{i_0} = j\}$.
So at least one of these sets is infinite; let it be $C = \{\beta \in A \mid \beta_{i_0} = j_0\}$. Set

$$C' = \{(\beta_1, \ldots, \beta_{i_0-1}, \beta_{i_0+1}, \ldots, \beta_n) \mid \beta = (\beta_1, \ldots, \beta_n) \in C\}.$$

But then also C' is an antichain and $|C'| = |C| = \infty$.
As $C' \subseteq \mathbb{N}^{n-1}$ we have reached a contradiction. □

Lemma 1.1.21 *Let $A \subseteq \mathbb{N}^n$ and set*

$$B = \{\beta \in A \mid \text{ there is no } \alpha \in A \text{ with } \alpha \prec \beta\}.$$

Then B is an antichain and for all $\alpha \in A$ there is a $\beta \in B$ with $\beta \preceq \alpha$.

Proof. Let $\beta, \beta' \in B$. Then we cannot have $\beta \prec \beta'$ or $\beta' \prec \beta$ by definition of B. Hence B is an antichain.

Let $\alpha \in A$. In the following way we define a sequence $\beta(0) \succ \beta(1) \succ \cdots$ with $\beta(i) \in A$. First of all, $\beta(0) = \alpha$. Secondly, if $\beta(i)$ is defined, for a certain $i \geq 0$, then $\beta(i+1)$ is any element of A with $\beta(i+1) \prec \beta(i)$, if such an element exists. The sequence stops if no such element exists in A. Note that the sequence has to be finite, because $|\beta(i+1)| < |\beta(i)|$. (Recall that $|\beta| = \sum_{i=1}^{n} \beta_i$.) Let $\beta(k)$ be the final element of the sequence. Then $\beta(k) \in B$ and $\beta(k) \preceq \alpha$. □

Lemma 1.1.22 (Dickson) *Let $A \subseteq \mathbb{N}^n$ e $I = \langle x^\alpha \mid \alpha \in A\rangle \subset R = k[x_1, \ldots, x_n]$. Then there exists an $A' \subset A$, with A' finite and $I = \langle x^\alpha \mid \alpha \in A'\rangle$.*

Proof. Define B as in Lemma 1.1.21. Then B is an antichain, hence finite (Lemma 1.1.20). Set $J = \langle x^\alpha \mid \alpha \in B\rangle$. We claim that $I = J$. The inclusion "$(\supseteq)$" is obvious because $B \subseteq A$. In order to show "$(\subseteq)$" let $\alpha \in A$. If $\alpha \in B$ then $x^\alpha \in J$. If $\alpha \notin B$ then by Lemma 1.1.21 there is a $\beta \in B$ with $\alpha \succeq \beta$. Then $\alpha = \beta + \gamma$ for a certain $\gamma \in \mathbb{N}^n$. But then $x^\alpha = x^\gamma x^\beta$, so that again $x^\alpha \in J$. We see that $x^\alpha \in J$ for all $\alpha \in A$, and therefore $I \subseteq J$. □

Theorem 1.1.23 *Let $\leq$ be an order on $\mathbb{N}^n$ with*

1) *$\leq$ is total,*

2) *if $\alpha \leq \beta$ then $\alpha + \gamma \leq \beta + \gamma$ for all $\gamma \in \mathbb{N}^n$,*

3) *$\alpha \geq \gamma_0 = (0, \ldots, 0)$ for all $\alpha \in \mathbb{N}^n$.*

Then $\leq$ is a monomial order.

Proof. We must show that there are no infinite descending chains. So suppose that we have $\alpha(1) \geq \alpha(2) \geq \ldots$. Let $A = \{\alpha(i) \mid i \geq 1\}$, and define B as in Lemma 1.1.21. Then B is an antichain, and hence finite (Lemma 1.1.20). Write $B = \{\beta(1), \ldots, \beta(r)\}$ with $\beta(1) > \beta(2) > \ldots > \beta(r)$. Let $\alpha \in A$. By Lemma 1.1.21 there is a $\beta(i) \in B$ with $\alpha \succeq \beta(i)$. Hence $\alpha = \beta(i) + \gamma$ for a certain $\gamma \in \mathbb{N}^n$. But then

$$\alpha = \beta(i) + \gamma \geq \beta(i) + \gamma_0 = \beta(i) \geq \beta(r).$$

So the descending chain has a minimal element, and is therefore finite. □

Theorem 1.1.24 (Hilbert's basis theorem) *Every ideal of $R = k[x_1, \ldots, x_n]$ is generated by a finite number of elements.*

PROOF. Let $I \subset R$ be an ideal and $J = \langle \mathrm{LM}(f) \mid f \in I \setminus \{0\} \rangle$ the ideal generated by the leading monomials of the elements of I (with respect to a fixed monomial order).

Dickson's lemma implies that there exist $g_1, \ldots, g_s \in I$ with

$$J = \langle \mathrm{LM}(g_i) \mid 1 \leq i \leq s \rangle.$$

These generate I. Indeed, let $f \in I$, then using polynomial division with remainder we find $r \in R$ such that there are $h_1, \ldots, h_s \in R$ with $f = h_1 g_1 + \ldots + h_s g_s + r$ and no monomial of r is divisible by an $\mathrm{LM}(g_i)$.

But $r = f - \sum h_i g_i \in I$ hence $\mathrm{LM}(r)$ lies in J. So there exists i such that $\mathrm{LM}(g_i)$ divides $\mathrm{LM}(r)$ (Lemma 1.1.18) but this is impossible except when $r = 0$. It follows that $f = \sum h_i g_i$. □

Corollary 1.1.25 *If I_k, for $k \geq 1$, are ideals of R with $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$ then there exists m with $I_m = I_{m+1} = I_{m+2} = \ldots$.*

PROOF. Let $J = \cup_{k \geq 1} I_k$ which is an ideal of R. Theorem 1.1.24 says that there are $g_1, \ldots, g_s \in J$ with $J = \langle g_1, \ldots, g_s \rangle$. Let k_i be such that $g_i \in I_{k_i}$ and set $m = \max(k_1, \ldots, k_s)$. Then $g_1, \ldots, g_s \in I_m$ and hence $I_{m+i} = I_m$ because $I_m \subseteq I_{m+i}$ is given and $I_{m+i} \subseteq J \subseteq I_m$. □

1.1.6 Gröbner bases

Here we fix a monomial order $\leq$ with respect to which we define the leading monomial of the elements of $k[x_1, \ldots, x_n]$.

Definition 1.1.26 *Let $I \subset R = k[x_1, \ldots, x_n]$ be an ideal. Then we write $\langle \mathrm{LM}(I) \rangle = \langle \mathrm{LM}(f) \mid f \in I \setminus \{0\} \rangle$. The monomial ideal $\langle \mathrm{LM}(I) \rangle$ is called the* initial ideal *of I.*

Definition 1.1.27 *Let $I \subset R = k[x_1, \ldots, x_n]$ be an ideal, and $G \subset I$ with $\langle \mathrm{LM}(I) \rangle = \langle \mathrm{LM}(g) \mid g \in G \setminus \{0\} \rangle$. Then G is called a* Gröbner basis *of I.*

Remark 1.1.28

- $G = I$ is a Gröbner basis of I.
- From Dickson's lemma (see also the proof of Hilbert's basis theorem) we immediately get that every ideal has a finite Gröbner basis.
- From the proof of Hilbert's basis theorem we see that a Gröbner basis of the ideal I generates I.

Lemma 1.1.29 *Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal. Then $G \subset I$ is a Gröbner basis of I if and only if for all $f \in I$, $f \neq 0$, there is a $g \in G$ such that $\mathrm{LM}(g)$ divides $\mathrm{LM}(f)$.*

PROOF. We have that G is a Gröbner basis of I if and only if

$$\langle \mathrm{LM}(g) \mid g \in G \rangle = \langle \mathrm{LM}(I) \rangle$$

so if and only if for all $f \in I$ there exists $g \in G$ such that $\mathrm{LM}(g)$ divides $\mathrm{LM}(f)$ (Lemma 1.1.18). □

Proposition 1.1.30 *Let $I \subset R = k[x_1, \ldots, x_n]$ be an ideal and $G \subset I$ a Gröbner basis. Let $f \in R$ and $r \in R$ a remainder of f modulo G. Then r is uniquely determined, that is, it does not depend on the choices made during the division algorithm. Moreover, $r = 0$ if and only if $f \in I$.*

PROOF. Suppose that

$$f = h_1 g_{i_1} + \ldots + h_s g_{i_s} + r_1 = \bar{h}_1 g_{j_1} + \ldots + \bar{h}_t g_{j_t} + r_2$$

with g_{i_k} and g_{j_k} in G and $r_i \in R$ with the property that none of their monomials are divisible by an $\mathrm{LM}(g)$ with $g \in G$.

Then

$$r_2 - r_1 = h_1 g_{i_1} + \ldots + h_s g_{i_s} - \bar{h}_1 g_{j_1} - \ldots - \bar{h}_t g_{j_t} \in I$$

so by Lemma 1.1.29 there exists $g \in G$ such that $\mathrm{LM}(g)$ divides $\mathrm{LM}(r_1 - r_2)$. But $\mathrm{LM}(r_1 - r_2)$ is a monomial of r_1 or of r_2 (or of both). Hence $r_2 - r_1 = 0$ so that $r_1 = r_2$.

If $f \in I$ then $r_1 \in I$ (because $r_1 = f - \sum_{u=1}^{s} h_u g_{i_u}$). So there exists $g \in G$ such that $\mathrm{LM}(g)$ divides $\mathrm{LM}(r_1)$ (Lemma 1.1.29) whence $r_1 = 0$. The other direction is trivial. □

1.1.7 Computing Gröbner bases

As before we write $R = k[x_1, \ldots, x_n]$ and we fix a monomial order $\leq$. Let $G \subset R \setminus \{0\}$.

Let $r \in R$ be a remainder of $f \in R$ modulo G. Then we write $r \stackrel{\circ}{=} \bar{f}^G$. So this means that there is a way to execute the division with remainder algorithm with input f and G so that the resulting remainder is r.

If $\bar{f}^G \stackrel{\circ}{=} 0$ then we say that f *reduces* to 0 modulo G.

Let $f_1, f_2 \in R \setminus \{0\}$ and write $x^{\alpha(i)} = \mathrm{LM}(f_i)$. Let γ be defined by $\gamma_j = \max(\alpha(1)_j, \alpha(2)_j)$. Then x^γ is called the *least common multiple* of $x^{\alpha(1)}$ and $x^{\alpha(2)}$.

Furthermore,

$$S(f_1, f_2) = \frac{x^\gamma}{\mathrm{LC}(f_1)\mathrm{LM}(f_1)} f_1 - \frac{x^\gamma}{\mathrm{LC}(f_2)\mathrm{LM}(f_2)} f_2$$

is called the *S-polynomial* of f_1 and f_2.

Example 1.1.31 Let $f_1 = 2xyz^2 - xy$ and $f_2 = 3x^2y^2z - 5xyz$. We use the order $<_{\mathrm{glex}}$. The least common multiple of the leading monomials of f_1 and f_2 is $x^\gamma = x^2y^2z^2$ and

$$S(f_1, f_2) = \frac{xy}{2}(2xyz^2 - xy) - \frac{z}{3}(3x^2y^2z - 5xyz) = -\frac{1}{2}x^2y^2 + \frac{5}{3}xyz^2,$$

hence $\mathrm{LM}(S(f_1, f_2)) = x^2y^2$.

We see that there is a remainder of $S(f_1, f_2)$ modulo $\{f_1, f_2\}$ which is not zero because x^2y^2 is not divisible by $\mathrm{LM}(f_1)$ or by $\mathrm{LM}(f_2)$.

In particular, we see that $\{f_1, f_2\}$ is not a Gröbner basis of the ideal generated by it.

An element of the form cx^α with $c \in k$ is called a *term*.

Lemma 1.1.32 *Let $G \subset R = k[x_1, \ldots, x_n]$ and $f \in R$. If f reduces to zero modulo G then we have $f = h_1 g_1 + \ldots + h_s g_s$ with $g_i \in G$ (not necessarily distinct) and where the $h_i \in R$ are terms with*

$$\mathrm{LM}(f) = \mathrm{LM}(h_1 g_1) > \mathrm{LM}(h_2 g_2) > \ldots > \mathrm{LM}(h_s g_s).$$

Proof. This follows from the division algorithm. □

The next theorem gives us a criterion with which we can decide whether a given set $G \subset R$ is a Gröbner basis (of the ideal it generates) or not.

Theorem 1.1.33 (Buchberger's criterion) *Let $I \subset R$ be an ideal generated by a set $G \subset I \setminus \{0\}$. Then G is a Gröbner basis of I if and only if $\overline{S(g_1, g_2)}^G \doteq 0$ for all $g_1, g_2 \in G$.*

Proof.

"⇒" If G is a Gröbner basis, then since $S(g_1, g_2) \in I$, this reduces to zero modulo G (Proposition 1.1.30).

"⇐" Let $f \in I \setminus \{0\}$; we show that there exists a $g \in G$ such that $\mathrm{LM}(g)$ divides $\mathrm{LM}(f)$. Then with Lemma 1.1.29 we conclude that G is a Gröbner basis of I.

We can write $f = h_1 g_1 + \ldots + h_s g_s$ where the $h_i \in R$ are terms and $g_i \in G$. Write $m_i = \mathrm{LM}(h_i g_i)$ and $m = m_1$. After reordering, we may assume that

$$m = m_1 = m_2 = \ldots = m_v > m_{v+1} \geq \ldots \geq m_s.$$

Let us also assume that $\mathrm{LC}(g_i) = 1$ (if $\mathrm{LC}(g_i)$ is not 1 then the same proof will go through, but the notation becomes a bit heavier).

Choose an expression of the form $f = h_1 g_1 + \ldots + h_s g_s$ with m minimal, and among the set of those expressions with v minimal.

Suppose that $v > 1$ and write $h_i = c_i x^{\alpha(i)}$. Then

$$\begin{aligned} f &= c_1 x^{\alpha(1)} g_1 + \ldots + c_s x^{\alpha(s)} g_s \\ &= c_1 (x^{\alpha(1)} g_1 - x^{\alpha(2)} g_2) + (c_1 + c_2) x^{\alpha(2)} g_2 + \ldots + c_s x^{\alpha(s)} g_s. \end{aligned}$$

Now let x^γ be the least common multiple of $\mathrm{LM}(g_1)$ and $\mathrm{LM}(g_2)$ and write $x^\delta = m = \mathrm{LM}(x^{\alpha(1)} g_1) = \mathrm{LM}(x^{\alpha(2)} g_2)$. Then

$$x^{\alpha(1)} g_1 - x^{\alpha(2)} g_2 = \frac{x^\delta}{\mathrm{LM}(g_1)} g_1 - \frac{x^\delta}{\mathrm{LM}(g_2)} g_2$$

because $x^\delta = x^{\alpha(i)} \mathrm{LM}(g_i)$.

Hence

$$x^{\alpha(1)} g_1 - x^{\alpha(2)} g_2 = x^{\delta - \gamma} \left(\frac{x^\gamma}{\mathrm{LM}(g_1)} g_1 - \frac{x^\gamma}{\mathrm{LM}(g_2)} g_2 \right) = x^{\delta - \gamma} S(g_1, g_2).$$

But $S(g_1, g_2)$ reduces to zero modulo G. Hence by Lemma 1.1.32

$$S(g_1, g_2) = u_1 g_{i_1} + \ldots + u_t g_{i_t}$$

where u_i is a term and

$$x^\gamma > \mathrm{LM}(S(g_1, g_2)) = \mathrm{LM}(u_1 g_{i_1}) > \mathrm{LM}(u_2 g_{i_2}) > \ldots > \mathrm{LM}(u_t g_{i_t}).$$

Therefore

$$x^{\alpha(1)} g_1 - x^{\alpha(2)} g_2 = \sum_{j=1}^{t} x^{\delta-\gamma} u_j g_{i_j}$$

and $\mathrm{LM}(x^{\delta-\gamma} u_j g_{i_j}) < x^\delta = m$. So since

$$f = c_1(x^{\alpha(1)} g_1 - x^{\alpha(2)} g_2) + (c_1 + c_2) x^{\alpha(2)} g_2 + \ldots + c_s x^{\alpha(s)} g_s$$

after subsituting we obtain an expression of the form $f = h_1 g_1 + \ldots + h_s g_s$ with v decreased, or in case $v = 2$ and $c_1 + c_2 = 0$, with m decreased. But this is impossible, hence $v = 1$.

It follows that $m = \mathrm{LM}(x^{\alpha(1)} g_1) = \mathrm{LM}(f)$ and hence $\mathrm{LM}(g_1)$ divides $\mathrm{LM}(f)$. □

Algorithm 1.1.34
Given: $g_1, \ldots, g_s$, all nonzero, generating the ideal $I \subset k[x_1, \ldots, x_n]$ and a fixed monomial order $\leq$.

We compute: a Gröbner basis of I with respect to $\leq$.

1. *Set $G_0 := \{g_1, \ldots, g_s\}$, and $i := 0$.*
2. *If $\overline{S(f,g)}^{G_i} \stackrel{\circ}{=} 0$ for all $f, g \in G_i$ then G_i is a Gröbner basis and we stop.*
3. *If there are $f, g \in G_i$ with $r \stackrel{\circ}{=} \overline{S(f,g)}^{G_i} \neq 0$ then we set $G_{i+1} := G_i \cup \{r\}$, $i := i + 1$, and return to 2.*

Proposition 1.1.35 *The algorithm 1.1.34 terminates correctly.*

Proof.

When the algorithm terminates G_i is a Gröbner basis of I because
- it generates I as it contains $g_1, \ldots, g_s$ and $G_i \subset I$,
- by Theorem 1.1.33 G_i is a Gröbner basis of the ideal it generates.

Now to termination. Consider the ideals $J_i = \langle \mathrm{LM}(g) \mid g \in G_i \rangle$. We claim that $G_{i+1} \supsetneq G_i$ implies that $J_{i+1} \supsetneq J_i$. Indeed, $G_{i+1} = G_i \cup \{r\}$ and $\mathrm{LM}(r)$ is not divisible by any $\mathrm{LM}(g)$ for $g \in G_i$. Hence $\mathrm{LM}(r) \notin J_i$ (Lemma 1.1.18). But $\mathrm{LM}(r) \in J_{i+1}$ so $J_{i+1} \supsetneq J_i$.

By Corollary 1.1.25 there are no infinite increasing chains of ideals in R. Hence the algorithm must terminate. □

Example 1.1.36 Let $g_1 = xyz - xy$ and $g_2 = x^2 y - yz$. We use the order $<_{\mathrm{glex}}$ with $x >_{\mathrm{glex}} y >_{\mathrm{glex}} z$.

We have $G_0 = \{g_1, g_2\}$ and

$$\begin{aligned} S(g_1, g_2) &= x g_1 - z g_2 \\ &= x^2 yz - x^2 y - (x^2 yz - yz^2) \\ &= -x^2 y + yz^2 \\ &\stackrel{g_2}{\to} yz^2 - yz \\ &= g_3. \end{aligned}$$

So $G_1 = \{g_1, g_2, g_3\}$. Next

$$\begin{aligned} S(g_1, g_3) &= zg_1 - xg_3 \\ &= xyz^2 - xyz - (xyz^2 - xyz) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} S(g_2, g_3) &= z^2 g_2 - x^2 g_3 \\ &= x^2yz^2 - yz^3 - (x^2yz^2 - x^2yz) \\ &= x^2yz - yz^3 \\ &\xrightarrow{g_1} -yz^3 + x^2y \\ &\xrightarrow{g_3} x^2y - yz^2 \\ &\xrightarrow{g_2} -yz^2 + yz \\ &\xrightarrow{g_3} 0. \end{aligned}$$

Hence $G_1 = \{g_1, g_2, g_3\}$ is a Gröbner basis of I.

When computing a Gröbner basis one can use a number of tricks that often make life easier. First of all, one can divide every element by a constant in order to make the leading coefficient equal to 1. A second trick is based on the following results.

Lemma 1.1.37 *Let $f, g \in k[x_1, \ldots, x_n]$ with $f, g \neq 0$, and assume that the least common multiple of $\mathrm{LM}(f)$ and $\mathrm{LM}(g)$ is the product $\mathrm{LM}(f)\mathrm{LM}(g)$ (so the greatest common divisor of $\mathrm{LM}(f)$ and $\mathrm{LM}(g)$ is 1). Let $u, v \in k[x_1, \ldots, x_n]$ be such that $\mathrm{LM}(u) < \mathrm{LM}(f)$ and $\mathrm{LM}(v) < \mathrm{LM}(g)$. Then $ug + vf$ reduces to 0 modulo $\{f, g\}$.*

Proof. The leading monomial of $ug + vf$ occurs in ug or in vf but not in both. Indeed, if $\mathrm{LM}(ug) = \mathrm{LM}(vf)$ then $\mathrm{LM}(u)\mathrm{LM}(g) = \mathrm{LM}(v)\mathrm{LM}(f)$. So since $\mathrm{LM}(g)$ and $\mathrm{LM}(f)$ have no common factors it follows that $\mathrm{LM}(g)$ divides $\mathrm{LM}(v)$. But $\mathrm{LM}(v) < \mathrm{LM}(g)$ so this is not possible. (Indeed, suppose that $\alpha > \beta$ and that x^α divides x^β. Then $\beta = \alpha + \gamma$ for a certain $\gamma \in \mathbb{N}^n$ with $\gamma \neq (0, 0, \ldots, 0)$. But then $\alpha > \alpha + \gamma > \alpha + \gamma + \gamma > \cdots$ and we have an infinite decreasing chain.)

It follows that $\mathrm{LM}(ug) \neq \mathrm{LM}(vf)$ so the biggest of the two is $\mathrm{LM}(ug + vf)$.

Suppose that $\mathrm{LM}(ug + vf) = \mathrm{LM}(ug)$. Hence $\mathrm{LM}(ug + vf)$ is divisible by $\mathrm{LM}(g)$ with factor $\mathrm{LM}(u)$. So in the division algorithm $ug + vf$ is replaced by

$$ug + vf - c\mathrm{LM}(u)g = (u - c\mathrm{LM}(u))g + vf$$

where $c \in k$.

We see that we have obtained an expression of the same form. So the division algorithm proceeds until it obtains zero. □

Lemma 1.1.38 *Let $f, g \in k[x_1, \ldots, x_n]$, $f, g \neq 0$, such that $\mathrm{LM}(f)$ and $\mathrm{LM}(g)$ have no common factors. Then $S(f, g)$ reduces to 0 modulo $\{f, g\}$.*

PROOF. Write $f = c_1x^\alpha + r_1$ and $g = c_2x^\beta + r_2$, with $x^\alpha = \mathrm{LM}(f)$, $x^\beta = \mathrm{LM}(g)$, $c_i \in k$. Then the least common multiple of x^α and x^β is $x^{\alpha+\beta}$. It follows that

$$\begin{aligned} S(f,g) &= \frac{x^{\alpha+\beta}}{c_1x^\alpha}(c_1x^\alpha + r_1) - \frac{x^{\alpha+\beta}}{c_2x^\beta}(c_2x^\beta + r_2) \\ &= \frac{1}{c_1}r_1x^\beta - \frac{1}{c_2}r_2x^\alpha \\ &= \frac{1}{c_1c_2}r_1(g - r_2) - \frac{1}{c_1c_2}r_2(f - r_1) \\ &= \frac{1}{c_1c_2}(r_1g - r_2f). \end{aligned}$$

Furthermore, $\mathrm{LM}(r_1) < \mathrm{LM}(f)$ and $\mathrm{LM}(r_2) < \mathrm{LM}(g)$. So by Lemma 1.1.37, $r_1g - r_2f$ reduces to zero modulo $\{f, g\}$. □

Conclusion. When calculating a Gröbner basis we don't need to check $S(f, g)$ if $\mathrm{LM}(f)$ and $\mathrm{LM}(g)$ don't have common factors.

Example 1.1.39 One problem with Gröbner bases is that on many occasions they are difficult to compute because they can be very big. As an example consider

$$f_1 = x^3 + y + z^2 - 1,\ f_2 = x^2 + y^3 + z - 1,\ f_3 = x + y^2 + z^3 - 1.$$

A Gröbner basis of the ideal generated by these polynomials with respect to $<_{\mathrm{lex}}$ is:

```
x + y + 7467678205870943/65719095621171465*z^25 +
        1906548593713883/4381273041411431*z^24 +
        8191728039103700/13143819124234293*z^23 -
        18118488190106417/65719095621171465*z^22 -
        40197573291173158/13143819124234293*z^21 -
        60670103256071060/13143819124234293*z^20 -
        152406006738162748/65719095621171465*z^19 +
        80624458839890252/13143819124234293*z^18 +
        649226374687018607/65719095621171465*z^17 +
        618150565337585002/65719095621171465*z^16 +
        353034549878420633/65719095621171465*z^15 +
        238802559571273606/65719095621171465*z^14 -
        11655150926515255/4381273041411431*z^13 -
        1972928184800696599/65719095621171465*z^12 -
        775940019593808291/21906365207057155*z^11 -
        1345970642704934506/65719095621171465*z^10 +
        2087861421527788238/65719095621171465*z^9 +
        3117371817891097726/65719095621171465*z^8 +
        374132655781594758/21906365207057155*z^7 -
        437890362723893518/65719095621171465*z^6 -
        60670429063906802/1991487746096105*z^5 -
        42413702095534382/65719095621171465*z^4 -
```

```
    28132496530919371/2190636520705715 5*z^3 +
    702615426148730872/65719095621171465*z^2 -
    80217895633896968/2190636520705715 5*z - 1,
y^2 - y - 7467678205870943/65719095621171465*z^25 -
    1906548593713883/4381273041411431*z^24 -
    8191728039103700/13143819124234293*z^23 +
    18118488190106417/65719095621171465*z^22 +
    40197573291173158/13143819124234293*z^21 +
    60670103256071060/13143819124234293*z^20 +
    152406006738162748/65719095621171465*z^19 -
    80624458839890252/13143819124234293*z^18 -
    649226374687018607/65719095621171465*z^17 -
    618150565337585002/65719095621171465*z^16 -
    353034549878420633/65719095621171465*z^15 -
    238802559571273606/65719095621171465*z^14 +
    11655150926515255/4381273041411431*z^13 +
    1972928184800696599/65719095621171465*z^12 +
    775940019593808291/21906365207057155*z^11 +
    134597064270493450 6/65719095621171465*z^10 -
    208786142152778823 8/65719095621171465*z^9 -
    311737181789109772 6/65719095621171465*z^8 -
    374132655781594758/21906365207057155*z^7 +
    437890362723893518/65719095621171465*z^6 +
    606704290639068 02/1991487746096105*z^5 +
    424137020955343 82/65719095621171465*z^4 +
    500388617379765 26/21906365207057155*z^3 -
    702615426148730872/65719095621171465*z^2 +
    80217895633896968/2190636520705715 5*z,
y*z + 1143517618732365 39/14896328340798865 4*z^25 +
    209291487801391117/44688985022396596 2*z^24 +
    52236360062652115/14896328340798865 4*z^23 -
    146842550247957727 4/22344492511198298 1*z^22 -
    858875628508250933/22344492511198298 1*z^21 -
    616034236519743323/22344492511198298 1*z^20 +
    879449288234641829 5/44688985022396596 2*z^19 +
    481569149010434716 9/44688985022396596 2*z^18 +
    114919988558291548 7/44688985022396596 2*z^17 -
    108878553866806735 6/22344492511198298 1*z^16 -
    251563354083999471 2/22344492511198298 1*z^15 +
    687051166591801069 9/44688985022396596 2*z^14 -
    333170034101035226 29/44688985022396596 2*z^13 -
    113874504830936685 2/74481641703994327*z^12 -
    240565993332030438 7/22344492511198298 1*z^11 +
    165321079713940029 75/14896328340798865 4*z^10 +
    384926380783824817 27/44688985022396596 2*z^9 -
    173059562480398345 78/22344492511198298 1*z^8 -
    304224353238662269 4/22344492511198298 1*z^7 -
```

```
      847056433871661220l/7448164170399432l*z^6 +
      109793303496803963/1194892647657663*z^5 -
      1834169915554595837/7448164170399432l*z^4 +
      23672654350351476275/446889850223965962*z^3 -
      2965819909779524184/7448164170399432l*z^2 +
      3212763377507509769/446889850223965962*z,
  z^26 - 9*z^23 + 29*z^20 - 6*z^18 - 11*z^17 - 14*z^16 + 27*z^15 - 110*z^14 +
      37*z^13 + 4*z^12 + 163*z^11 + 36*z^10 - 173*z^9 + 28*z^8 - 146*z^7 +
      208*z^6 - 94*z^5 + 89*z^4 - 91*z^3 + 37*z^2 - 5*z
```

1.2 Applications of Gröbner bases

In this second section we see a few applications of Gröbner bases. Of course, they solve the main problem: given an ideal $I \subset k[x_1, \ldots, x_n]$ and $f \in k[x_1, \ldots, x_n]$ decide if $f \in I$. In order to solve this problem we compute a Gröbner basis G of I. Then $f \in I$ if and only if $\bar{f}^G = 0$.

1.2.1 Solving polynomial equations

Definition 1.2.1 *A field k is called* algebraically closed *if every polynomial in $k[x]$ has a zero in k.*

Example 1.2.2 For example $k = \mathbb{C}$.

The problem of this section is, given $f_1, \ldots, f_s$ in $k[x_1, \ldots, k_n]$, to decide if there is a vector $(a_1, \ldots, a_n) \in k^n$ with

$$f_1(a_1, \ldots, a_n) = \ldots = f_s(a_1, \ldots, a_n) = 0$$

and in the affirmative case to find such $(a_1, \ldots, a_n)$.

If the field is algebraically closed the first part of the problem can be solved with Hilbert's Nullstellensatz.

Theorem 1.2.3 (Hilbert's Nullstellensatz) *Let k be an algebraically closed field and $R = k[x_1, \ldots, x_n]$. Let $f_1, \ldots, f_s \in R$. Then there exists $\bar{a} = (a_1, \ldots, a_n) \in k^n$ with $f_1(\bar{a}) = \ldots = f_s(\bar{a}) = 0$ if and only if the ideal $I = \langle f_1, \ldots, f_s \rangle$ does not contain 1.*

Here we give a recent proof of this famous theorem, due to Lev Glebsky ("A proof of Hilbert's Nullstellensatz based on Groebner bases", preprint, 2012). This proof uses Gröbner bases. First we show a few lemmas. For these we do not assume that k is necessarily algebraically closed, unless otherwise stated.

Lemma 1.2.4 *Let I, J be ideals of R. Let t be an extra indeterminate. Consider the ideal M of $k[t, x_1, \ldots, x_n]$ generated by all tf and $(1-t)g$ for $f \in I$ and $g \in J$. Then $I \cap J = M \cap R$.*

Proof. First of all, note that M consists of elements of the form

$$h_1 t f_1 + \cdots + h_r t f_r + k_1(1-t)g_1 + \cdots + k_s(1-t)g_s \text{ where } h_i, k_j \in k[t, x_1, \ldots, x_n],\ f_i \in I,\ g_j \in J. \tag{1.1}$$

Let $h \in M \cap R$. Then $h \in M$ so h can be written as in (1.1). But also $h \in R$, so h is independent of t, which means that substituting a value $a \in k$ for t leaves h unchanged. Now setting $t = 1$ shows $h = \sum_i h_i(1, x_1, \ldots, x_n) f_i \in I$ and setting $t = 0$ proves $h = \sum_j k_j(0, x_1, \ldots, x_n) g_j \in J$.

Conversely, let $h \in I \cap J$. Then $h = th + (1-t)h \in M$. So $h \in M \cap R$. □

Later we will see how this lemma leads to an algorithm for computing generators of the intersection of two ideals.

In the next lemma we consider polynomials in x_1. If $f \in k[x_1]$ is such a polynomial, and $I \subset R$ an ideal then we write

$$\langle f \rangle + I = \{hf + g \mid h \in R, g \in I\}$$

which is an ideal of R.

Lemma 1.2.5 *Let $I \subset R$ be an ideal. Let $f_1, f_2 \in k[x_1]$ be such that $\gcd(f_1, f_2) = 1$. Set $I_i = \langle f_i \rangle + I$. Then $I_1 \cap I_2 = \langle f_1 f_2 \rangle + I$.*

Proof. Let $g_1, \ldots, g_s \in R$ be such that $I = \langle g_1, \ldots, g_s \rangle$. Then $I_i = \langle f_i, g_1, \ldots, g_s \rangle$. Let t be an extra indeterminate. Let J be the ideal generated by tf_1, $(1-t)f_2$, tg_i, $(1-t)g_i$ for $1 \leq i \leq s$. So J is also generated by tf_1, $(1-t)f_2$, $g_1, \ldots, g_s$.

Now let $h_1, h_2 \in k[x_1]$ be such that $h_1 f_1 + h_2 f_2 = 1$. Then J is also generated by $f_1 f_2$, $h_2 f_2 - t$, $g_1, \ldots, g_s$. Indeed let J' be the ideal generated by the latter polynomials. Since $tf_1 = f_1 f_2 h_2 - (h_2 f_2 - t) f_1$ and $(1-t)f_2 = f_1 f_2 h_1 + (h_2 f_2 - t) f_2$ we get $J \subset J'$. But also $J' \subset J$ as $f_1 f_2 = (tf_1) f_2 + ((1-t)f_2) f_1$ and $h_2 f_2 - t = ((1-t)f_2) h_2 - t f_1 h_1$.

We claim that $J \cap R = \langle f_1 f_2, g_1, \ldots, g_s \rangle$. The inclusion $\supset$ is obvious. For the other one, let $g \in J \cap R$, then $g = a f_1 f_2 + b(h_2 f_2 - t) + \sum_i c_i g_i$ for certain $a, b, c_i \in k[t, x_1, \ldots, x_n]$. Since $g \in R$, the subsitution $t \mapsto h_2 f_2$ leaves g unchanged. But it also maps g to $\bar{a} f_1 f_2 + \sum_i \bar{c}_i g_i$ (where $\bar{a}$, $\bar{c}_i$ is the image of a, c_i under the subsitution). It follows that we also have $\subset$.

Now Lemma 1.2.4 finishes the proof. □

Let $a_1, \ldots, a_s \in k$. Then we define the ring homomorphism $\mathrm{ev}_{a_1,\ldots,a_s} : R \to k[x_{s+1}, \ldots, x_n]$ by $\mathrm{ev}_{a_1,\ldots,a_s}(f) = f(a_1, \ldots, a_s, x_{k+1}, \ldots, x_n)$.

Lemma 1.2.6 *Assume that k is algebraically closed. Let $I \subset R$ be a proper ideal (i.e., $I \neq R$). Then there is $a \in k$ such that $\mathrm{ev}_a(I)$ is a proper ideal of $k[x_2, \ldots, x_n]$.*

Proof. Set $J = I \cap k[x_1]$. This is an ideal of $k[x_1]$. It cannot be all of $k[x_1]$ as otherwise $1 \in I$ and $I = R$. We distinguish two cases. In the **first** case $J \neq 0$, so that $J = \langle p \rangle$ for some $p \in k[x_1]$ (Lemma 1.1.5). Write $p = (x_1 - a_1)^{m_1} \cdots (x_1 - a_r)^{m_r}$, where the $a_i \in k$ are pairwise distinct. Now using Lemma 1.2.5 we get

$$I = \langle p \rangle + I = \bigcap_{i=1}^{r} (\langle (x_1 - a_i)^{m_i} \rangle + I).$$

So there is an i such that $\langle (x_1 - a_i)^{m_i} \rangle + I \neq R$. Denote this ideal by J_i and write $M = \langle x_1 - a_i \rangle + I$. Now

$$\langle (x_1 - a_i)^{m_i} \rangle + I \subset M \subset \sqrt{J_i}.$$

(For the definition of $\sqrt{J_i}$ see Definition 1.2.12). But also $\sqrt{J_i} \neq R$, whence $M \neq R$. But M has a generating set consisting of $x_1 - a_i$, $g_1, \ldots, g_s$, with $g_i \in k[x_2, \ldots, x_n]$. It follows that $1 \notin \mathrm{ev}_{a_i}(M)$, and therefore $1 \notin \mathrm{ev}_{a_i}(I)$ (as $I \subset M$).

In the **second** case $J = 0$. Now we consider the rational function field $k(x_1)$ and the polynomial ring $S = k(x_1)[x_2, \ldots, x_n]$. Then R can be viewed as a subset of S. Let I' be the ideal of S generated by I. Then $J = 0$ implies that $I' \neq S$, i.e., I' is a proper ideal. Let G' be a Gröbner basis of I', consisting of monic elements. The coefficients of the elements of G' lie in $k(x_1)$, so are of the form p/q with $p, q \in k[x_1]$. Let h be the product of all denominators that occur in the coefficients of the elements of G'. Since k is infinite, there is $a \in k$ with $h(a) \neq 0$. This means that $\mathrm{ev}_a(g)$ is a well defined element of $k[x_2, \ldots, x_n]$ for all $g \in G'$. We have that $\mathrm{ev}_a(G')$ is a Gröbner basis of the ideal M of $k[x_2, \ldots, x_n]$ generated by it. (Indeed, let $f = \sum_i h_i \mathrm{ev}_a(g_i) \in M$ with $h_i \in k[x_2, \ldots, x_n]$; we must show that there is a $g' \in \mathrm{ev}_a(G')$ such that $\mathrm{LM}(g')$ divides $\mathrm{LM}(f)$. Consider $\tilde{f} = \sum_i h_i g_i \in I'$, and note that $\mathrm{ev}_a(\tilde{f}) = f$. If $\mathrm{LM}(f) = \mathrm{LM}(\tilde{f})$ then we are done. Otherwise $\mathrm{ev}_a(\mathrm{LC}(\tilde{f})) = 0$. On the other hand, there is a $g \in G'$ such that $\mathrm{LM}(g)$ divides $\mathrm{LM}(\tilde{f})$. We now replace $\tilde{f}$ by $\hat{f} = \tilde{f} - cx^\alpha g$, where $c \in k(x_1)$ is such that $\mathrm{ev}_a(cx^\alpha) = 0$. Also $\mathrm{ev}_a(\hat{f}) = f$, but $\mathrm{LM}(\hat{f}) < \mathrm{LM}(\tilde{f})$. Continuing like this we eventually find an $\bar{f} \in I'$ such that $\mathrm{ev}_a(\bar{f}) = f$ and $\mathrm{LM}(\bar{f}) = \mathrm{LM}(f)$.)

However, $\mathrm{ev}_a(G')$ does not contain any constants, as $J = 0$. Therefore $M \neq k[x_2, \ldots, x_n]$. So since $\mathrm{ev}_a(I) \subset M$, also $\mathrm{ev}_a(I) \neq k[x_2, \ldots, x_n]$. □

Proof.(of the Nullstellensatz). The proof is by induction on n, the case $n = 1$ being clear. If $n > 1$ then by Lemma 1.2.6 we get that there is an $a_1 \in k$ such that $\mathrm{ev}_{a_1}(I)$ is a proper ideal of $k[x_2, \ldots, x_n]$. Hence by induction there are $a_2, \ldots, a_n \in k$ such that $f(a_1, a_2, \ldots, a_n) = 0$ for all $f \in I$. □

The work of David Hilbert (1862 - 1943)

is of great importance for the development and use of Gröbner bases. However, interestingly, his methods were not algorithmical at all. His proof of what is now known as Hilbert's basis theorem shows that every ideal has a finite number of generators, without giving a clue as to what these generators are, or how many are needed. This prompted Gordan, famously, to exclaim "Das ist nicht Mathematik. Das ist Theologie." (This is not mathematics. This is theology.)

Using the Nullstellensatz we can decide whether or not a given set of polynomial equations has a solution in an algebraically closed field. Indeed, we compute a Gröbner basis G of the ideal generated by the polynomials. Then 1 lies in the ideal if and only if $\bar{1}^G = 0$.

Now we turn to the problem of actually determining a solution, if one exists. The following lemma is trivial, but nonetheless important.

Lemma 1.2.7 *Let I be the ideal generated by $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$, and let $\bar{a} = (a_1, \ldots, a_n) \in k^n$. Then $f_1(\bar{a}) = \ldots = f_s(\bar{a}) = 0$ if and only if $f(\bar{a}) = 0$ for all $f \in I$.*

PROOF. Note that $f \in I$ if and only if there exist $h_1, \ldots, h_s \in R$ with $f = h_1 f_1 + \cdots + h_s f_s$. □

From this lemma we see that solving the polynomial equations $f_1 = \ldots = f_s = 0$ is equivalent to solving the equations $g_1 = \ldots = g_t = 0$, where $\{g_1, \ldots, g_t\}$ is any generating set of the ideal generated by the f_i. Next we see that Gröbner bases with respect to the lexicographical order are particularly useful in this respect.

Definition 1.2.8 *Let $I \subset k[x_1, \ldots, x_n]$ be an ideal and $0 \leq l \leq n-1$. Then $I \cap k[x_{l+1}, \ldots, x_n]$, which is an ideal of $k[x_{l+1}, \ldots, x_n]$, is called the* l-th elimination ideal *of I.*

The l-th elimination ideal eliminates the first l indeterminates.

Theorem 1.2.9 (Elimination theorem) *Set $R = k[x_1, \ldots, x_n]$, and use the order $<_{\text{lex}}$ with $x_1 >_{\text{lex}} x_2 >_{\text{lex}} \ldots >_{\text{lex}} x_n$. Let $I \subset R$ be an ideal and G a Gröbner basis of I with respect to $<_{\text{lex}}$. Then $G \cap k[x_{l+1}, \ldots, x_n]$ is a Gröbner basis of $I \cap k[x_{l+1}, \ldots, x_n]$.*

PROOF. Set $G_l = G \cap k[x_{l+1}, \ldots, x_n]$ and $I_l = I \cap k[x_{l+1}, \ldots, x_n]$.

We need to show that $\langle \text{LM}(I_l) \rangle = \langle \text{LM}(g) \mid g \in G_l \rangle$.

"$\supset$" Obvious as $G_l \subset I_l$.

"$\subset$" Let $f \in I_l$. Then $f \in I$ and since G is a Gröbner basis of I, there exists $g \in G$ such that $\text{LM}(g)$ divides $\text{LM}(f)$. But $f \in k[x_{l+1}, \ldots, x_n]$ so also $\text{LM}(g) \in k[x_{l+1}, \ldots, x_n]$.

Now let x^α be a monomial of g. If $\alpha_i > 0$ for some $i \leq l$ then $x^\alpha >_{\text{lex}} \text{LM}(g)$. But that is impossible.

Hence $x^\alpha \in k[x_{l+1}, \ldots, x_n]$ and therefore $g \in k[x_{l+1}, \ldots, x_n]$. It follows that $g \in G_l$ whence $\text{LM}(f) \in \langle \text{LM}(g) \mid g \in G_l \rangle$. □

Conclusion. With a Gröbner basis with respect to $<_{\text{lex}}$ we immediately find Gröbner bases for

$$\begin{aligned} I &\cap k[x_n] \\ I &\cap k[x_{n-1}, x_n] \\ &\vdots \\ I &\cap k[x_2, \ldots, x_n] \\ I &\cap k[x_1, \ldots, x_n] = I. \end{aligned}$$

So the Gröbner basis has a triangular structure that may help to solve the corresponding polynomial equations.

Note: it is possible that $I \cap k[x_{l+1}, \ldots, x_n] = 0$.

Example 1.2.10 Consider the polynomials

$$f_1 = x^2 + y + z - 1, \quad f_2 = x + y^2 + z - 1, \quad f_3 = x + y + z^2 - 1.$$

We want to solve the system $f_1 = f_2 = f_3 = 0$.

A Gröbner basis with respect to $>_{\text{lex}}$ consists of the polynomials

$$\begin{aligned} g_1 &= x + y + z^2 - 1 \\ g_2 &= y^2 - y - z^2 + z \\ g_3 &= 2yz^2 + z^4 - z^2 \\ g_4 &= z^6 - 4z^4 + 4z^3 - z^2 = z^2(z-1)^2(z^2 + 2z - 1). \end{aligned}$$

With $I = \langle f_1, f_2, f_3 \rangle$ we see that $I \cap k[z]$ is generated by g_4, $I \cap k[y, z]$ is generated by g_2, g_3, g_4.

Also we see that $g_4 = 0$ gives a finite number of values of z, namely 0, 1 and $-1 \pm \sqrt{2}$. If $z = 0$ then also $g_3 = 0$, but $g_2 = 0$ implies that y can be 0 or 1. If $y = z = 0$, then from $g_1 = 0$ it follows that $x = 1$. We find the solution $(1, 0, 0)$. Going on like this we find all solutions:

$$\begin{aligned} &(1, 0, 0) \\ &(0, 1, 0) \\ &(0, 0, 1) \\ &(-1 - \sqrt{2}, -1 - \sqrt{2}, -1 - \sqrt{2}) \\ &(-1 + \sqrt{2}, -1 + \sqrt{2}, -1 + \sqrt{2}). \end{aligned}$$

Example 1.2.11 Theorem 1.2.9 and Lemma 1.2.4 give an algorithm for computing generators of the intersection of two ideals $I = \langle f_1, \ldots, f_r \rangle$, $J = \langle g_1, \ldots, g_s \rangle$. Indeed, let t be an extra indeterminate, and

$$M = \langle tf_1, \ldots, tf_r, (1-t)g_1, \ldots, (1-t)g_s \rangle,$$

which is an ideal of $k[t, x_1, \ldots, x_n]$. Let G be a Gröbner basis of M with respect to a lexicographical ordering, such that $t >_{\text{lex}} x_i$ for all i. Then $G \cap k[x_1, \ldots, x_n]$ is a Gröbner basis of $I \cap J$.

1.2.2 Applications of Gröbner bases in geometry

Definition 1.2.12 *Let $I \subset k[x_1, \ldots, x_n]$ be an ideal. Then*

$$\sqrt{I} = \{ f \in k[x_1, \ldots, x_n] \mid \text{ there is } m > 0 \text{ with } f^m \in I \}$$

is called the radical *of I.*

Remark 1.2.13 $\sqrt{I}$ is an ideal of $k[x_1, \ldots, x_n]$.

Proof. Obviously $0 \in \sqrt{I}$. Let $f \in \sqrt{I}$ and $g \in k[x_1, \ldots, x_n]$, so $f^m \in I$ for a certain $m > 0$. Hence $g^m f^m = (gf)^m \in I$ so that $gf \in \sqrt{I}$.

If $f, g \in \sqrt{I}$ then we let $m > 0$ be such that $f^m, g^m \in I$. Then

$$(f+g)^{2m} = \sum_{i=0}^{2m} \binom{2m}{i} f^i g^{2m-i} \in I$$

so $f + g \in \sqrt{I}$. □

Lemma 1.2.14 *Let $I = \langle f_1, \ldots, f_s \rangle \subset k[x_1, \ldots, x_n]$ be an ideal and $f \in k[x_1, \ldots, x_n]$. Then $f \in \sqrt{I}$ if and only if 1 is contained in the ideal $J = \langle f_1, \ldots, f_s, 1 - yf \rangle \subset k[x_1, \ldots, x_n, y]$.*

PROOF. "$\Rightarrow$" Suppose that $f^m \in I$. Then $y^m f^m \in J$ (because $I \subset J$). But also

$$(1 - y^m f^m) = (1 - yf)(1 + yf + (yf)^2 + \ldots + (yf)^{m-1}) \in J$$

so $1 \in J$.

"$\Leftarrow$" If $1 \in J$ then there exist $p_i, q \in k[x_1, \ldots, x_n, y]$ with

$$1 = \sum_{i=1}^{s} p_i(x_1, \ldots, x_n, y) f_i + q(x_1, \ldots, x_n, y)(1 - yf).$$

Now we formally subsitite $\frac{1}{f}$ for y and get

$$1 = \sum_{i=1}^{s} p_i(x_1, \ldots, x_n, \frac{1}{f}) f_i.$$

Note that the $p_i(x_1, \ldots, x_n, \frac{1}{f})$ are rational expressions with denominators f^r for certain $r \geq 0$.

Hence there is an m with $f^m p_i(x_1, \ldots, x_n, \frac{1}{f}) \in k[x_1, \ldots, x_n]$ and therefore

$$f^m = \sum_{i=1}^{s} f^m p_i(x_1, \ldots, x_n, \frac{1}{f}) f_i$$

so $f^m \in I$, in other words $f \in \sqrt{I}$. □

So with Gröbner bases one can check whether $f \in \sqrt{I}$ or not. This leads to the following algorithm.

Algorithm 1.2.15
Given: $f_1, \ldots, f_s$ generating the ideal $I \subset k[x_1, \ldots, x_n]$, and an $f \in k[x_1, \ldots, x_n]$.
We decide $f \in \sqrt{I}$ or not.

1. *Compute a Gröbner basis G of $J = \langle f_1, \ldots, f_s, 1 - yf \rangle$;*
2. *If $\bar{1}^G = 0$ then $f \in \sqrt{I}$, otherwise $f \notin \sqrt{I}$.*

Definition 1.2.16 *Let $W = k^n$ be a vector space over k of dimension n. Let $I = \langle f_1, \ldots, f_s \rangle$ be an ideal of $k[x_1, \ldots, x_n]$. Then the set*

$$V(I) = \{(w_1, \ldots, w_n) \in W \mid f(w_1, \ldots, w_n) = 0 \text{ for all } f \in I\}$$

is called the closed set *corresponding to I.*

Theorem 1.2.17 (Hilbert's Nullstellensatz, strong form) *Let k be an algebraically closed field and $I \subset k[x_1, \ldots, x_n]$ an ideal. Let $J \subset k[x_1, \ldots, x_n]$ be the ideal defined by*

$$J = \{f \in k[x_1, \ldots, x_n] \mid f(w) = 0, \ \forall w \in V(I)\}.$$

Then $J = \sqrt{I}$.

PROOF. If $f \in \sqrt{I}$ then there exists $m > 0$ with $f^m \in I$ and hence $f^m(w) = 0$ for all $w \in V(I)$. But $f^m(w) = (f(w))^m$ so $f(w) = 0$ for all $w \in V(I)$. Therefore $f \in J$.

Now take $f \in J$, that is, $f(w) = 0$ for all $w \in V(I)$. Let I be generated by $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$. Set $\tilde{I} = \langle f_1, \ldots, f_s, 1 - yf \rangle \subset k[x_1, \ldots, x_n, y]$. Let $w = (w_1, \ldots, w_n, w_{n+1}) \in k^{n+1}$. We consider two cases:

- $(w_1, \ldots, w_n) \in V(I)$. Then $f_i(w) = 0$ for $i = 1, \ldots, s$. But $f(w) = 0$ so $(1 - yf)(w) \neq 0$.

- $(w_1, \ldots, w_n) \notin V(I)$. Then there is f_i with $f_i(w) \neq 0$.

The conclusion is that $\{w = (w_1, \ldots, w_{n+1}) \mid h(w) = 0,\ \forall h \in \tilde{I}\} = \emptyset$. By the weak form of Hilbert's Nullstellensatz (Theorem 1.2.3), $1 \in \tilde{I}$ (k is algebraically closed). Hence by Lemma 1.2.14, $f \in \sqrt{I}$. □

Remark 1.2.18 One has

$$V(I) \cap V(J) = V(I + J)$$

where $I + J$ is the ideal

$$I + J = \{f + g \mid f \in I, g \in J\}.$$

PROOF. Let $v \in V(I) \cap V(J)$ and $h = f + g \in I + J$ with $f \in I$ and $g \in J$. Then

$$h(v) = f(v) + g(v) = 0 + 0 = 0$$

so $v \in V(I + J)$.

Conversely, let $v \in V(I + J)$. Since $I \subset I + J$ we get $h(v) = 0$ for all $h \in I$, whence $v \in V(I)$. Analogously we get $v \in V(J)$. Therefore $v \in V(I) \cap V(J)$. □

From this it follows that for $X \subset k^n$ there exists a unique minimal closed set containing X. This is called the *closure* of X. Now let

$$\mathrm{Id}(X) = \{f \in k[x_1, \ldots, x_n] \mid f(v) = 0,\ \forall v \in X\}$$

which is an ideal of $k[x_1, \ldots, x_n]$. Then the closure of X is $V(\mathrm{Id}(X))$. (Indeed, let J be such that the closure of X is $V(J)$. Let $f \in J$, then $f(v) = 0$ for all $v \in X$, whence $f \in \mathrm{Id}(X)$. So $J \subset \mathrm{Id}(X)$ implying $V(\mathrm{Id}(X)) \subset V(J)$. But $V(\mathrm{Id}(X))$ is a closed set containing X, so we also get the reverse inclusion.)

Theorem 1.2.19 *Let k be an algebraically closed field, and $R = k[x_1, \ldots, x_n]$, and $I = \langle f_1, \ldots, f_s \rangle \subset R$. Let $\pi_l \colon k^n \to k^{n-l}$ be defined by $\pi_l(v_1, \ldots, v_n) = (v_{l+1}, \ldots, v_n)$.*

Let $I_l = k[x_{l+1}, \ldots, x_n] \cap I$, the l-th elimination ideal of I. Then $V(I_l)$ is the closure of $\pi_l(V(I))$.

Example 1.2.20 Let $I = \langle xy - 1 \rangle \subset k[x, y]$. Consider $\pi_1 \colon k^2 \to k$ with $\pi_1(v_1, v_2) = v_2$. Then

$$V(I) = \{(v_1, v_2) \mid v_1 v_2 = 1\} \quad \Longrightarrow \quad \pi_1(V(I)) = k \setminus \{0\}.$$

Now $I_1 = 0$ which implies $V(I_1) = k$.

PROOF.(of Theorem 1.2.19) Let $J = \mathrm{Id}(\pi_l(V(I)) \subset k[x_{l+1}, \ldots, x_n]$. Then we claim that $J = \sqrt{I_l}$. In order to see "$\supset$", let $f \in \sqrt{I_l}$. Then $f^m \in I_l \subset I$, for some $m > 0$. So, for $v \in V(I)$ we have $f^m(v) = 0$, and consequently $f(v) = 0$. It follows that $f \in J$.

For the reverse inclusion let $f \in J$. Then $f(v_{l+1}, \ldots, v_n) = 0$ for all $(v_1, \ldots, v_n) \in V(I)$. It follows that $f(v) = 0$ for all $v \in V(I)$ and hence, by Hilbert's Nullstellensatz (Theorem 1.2.17), $f \in \sqrt{I}$. In other words, $f^m \in I$. But $f \in k[x_{l+1}, \ldots, x_n]$ so f^m also lives there. Therefore $f^m \in I \cap k[x_{l+1}, \ldots, x_n] = I_l$.

We conclude that $V(J) = V(\sqrt{I_l})$. But the latter is equal to $V(I_l)$. □

Conclusion. In view of the elimination theorem, using Gröbner bases we can find the closure of $\pi_l(U)$, where $U \subset k^n$ is a closed set.

As an application we show how to find the closure of the image of a regular function.

A *regular function* $h\colon k^n \to k^m$ is given by m polynomials $h_1, \ldots, h_m \in k[x_1, \ldots, x_n]$, such that $h(v) = (h_1(v), \ldots, h_m(v))$.

Let $X = V(I) \subset k^n$ where $I = \langle f_1, \ldots, f_s \rangle \subset k[x_1, \ldots, x_n]$. We want to find the closure of $h(X)$.

Let

$$\Gamma = \{(v, h(v)) \mid v \in X\} \subset k^{n+m}.$$

This is called the *graph* of h.

Observe that $\Gamma \subset k^{n+m}$ is closed. Indeed, let J be the ideal of $k[x_1, \ldots, x_n, y_1, \ldots, y_m]$ generated by

$$\{f_1, \ldots, f_s, y_1 - h_1, \ldots, y_m - h_m\}.$$

Then $(v, w) \in \Gamma$ if and only if $p(v, w) = 0$ for all $p \in J$ ($w = h(v)$ if and only if $(y_i - h_i)(v, w) = w_i - h_i(v) = 0$).

By computing a Gröbner basis we can compute the closure of $\pi_n(\Gamma)$ where $\pi_n\colon k^{n+m} \to k^m$ is defined by

$$\pi_n(v_1, \ldots, v_n, w_1, \ldots, w_m) = (w_1, \ldots, w_m).$$

Now $\pi_n(\Gamma) = h(X)$ so we see that we can compute the closure of $h(X)$ (here we always assume the field k to be algebraically closed).

Example 1.2.21 Let $S = \{(t^2, t^3) \mid t \in \mathbb{C}\}$. We compute the closure of S. Define $h\colon \mathbb{C} \to \mathbb{C}^2$ by $h(t) = (t^2, t^3)$. Then $S = \{h(t) \mid t \in \mathbb{C}\}$. Consider the graph of h:

$$\Gamma = \{(u, h(u)) \mid u \in \mathbb{C}\}.$$

We have that $\Gamma = V(I)$ where $I = \langle y_1 - t^2, y_2 - t^3 \rangle \subset k[t, y_1, y_2]$.

We compute a Gröbner basis of I with respect to $<_{\mathrm{lex}}$ with $t >_{\mathrm{lex}} y_1 >_{\mathrm{lex}} y_2$, and obtain

$$G = \{t^2 - y_1, ty_1 - y_2, ty_2 - y_1^2, y_1^3 - y_2^2\}.$$

So by the elimination theorem, a Gröbner basis of $I \cap k[y_1, y_2]$ is $G \cap k[y_1, y_2] = \{y_1^3 - y_2^2\}$. The conclusion is that the closure of S is $V(J)$ where J is generated by $y_1^3 - y_2^2$.

Here we have $V(J) = S$, but this is not true in general.

The set S is the parametric form of a curve in $\mathbb{C}^2$, whereas the representation as $V(J)$ is called the implicit form of the curve. It is not true in general that the parametric form and the implicit form define exactly the same set of points.

Divertimento: the real Nullstellensatz

In general the Nullstellensatz does not hold when the field is not algebraically closed. However, for the field is $\mathbb{R}$ there is an interesting variant of the theorem. It is called the *real Nullstellensatz*, (see G. Stengle. A Nullstellensatz and a Positivestellensatz in semi-algebraic geometry, Math.Ann.(1974),pp. 87-97).

For this let I be an ideal of $R = \mathbb{R}[x_1, \ldots, x_n]$. Then its real radical is defined as

$$\sqrt[\mathbb{R}]{I} = \{f \in R \mid f^{2m} + \sum_i r_i^2 \in I \text{ for some } m > 0, r_i \in R\}.$$

With this definition the real Nullstellensatz holds: let J be as in the strong form of the Nullstellensatz, then

$$J = \sqrt[\mathbb{R}]{I}.$$

Example: $I = \langle x_1^2 + x_2^2 \rangle$; then it is not difficult to see that

$$\sqrt[\mathbb{R}]{I} = \langle x_1, x_2 \rangle.$$

1.2.3 An application in combinatorics: Alon's non-vanishing theorem

Here we let k be a field. Let $T_1, \ldots, T_n$ be finite subsets of k, and write $t_i = |T_i|$. Then we form the set of points $T = T_1 \times T_2 \times \cdots \times T_n \subset k^n$. We set

$$I(T) = \{f \in k[x_1, \ldots, x_n] \mid f(p) = 0 \text{ for all } p \in T\}.$$

It is straightforward to see that $I(T)$ is an ideal of $k[x_1, \ldots, x_n]$. For $1 \leq i \leq n$ set

$$f_i = \prod_{s \in T_i} (x_i - s).$$

These are elements of $k[x_i]$, and obviously lie in $I(T)$.

Lemma 1.2.22 *$\{f_1, \ldots, f_n\}$ is a Gröbner basis of $I(T)$ (with respect to any monomial ordering).*

Proof. The leading monomial of f_i is $x_i^{t_i}$. Pairwise they have no common factors; therefore the f_i form a Gröbner basis of the ideal they generate.

We need to show that the f_i generate $I(T)$. For that we use induction on n. For $n = 1$ it is obvious, so suppose $n > 1$.

Let $f \in I(T)$ and write $f = g_0 + g_1 x_n + \cdots + g_r x_n^r$, with $g_i \in k[x_1, \ldots, x_{n-1}]$. Suppose that $f \notin \langle f_1, \ldots, f_n \rangle$, and choose f with this property such that r is minimal.

First suppose that $g_i(u_1, \ldots, u_{n-1}) = 0$ for all i and $(u_1, \ldots, u_{n-1}) \in T_1 \times \cdots \times T_{n-1}$. Then by induction the g_i lie in the ideal $\langle f_1, \ldots, f_{n-1} \rangle \subset k[x_1, \ldots, x_{n-1}]$. Hence $f \in \langle f_1, \ldots, f_{n-1} \rangle \subset k[x_1, \ldots, x_n]$. So we obtain a contradiction.

Now assume that there is a $(u_1, \ldots, u_{n-1}) \in T_1 \times \cdots \times T_{n-1}$ with $g_j(u_1, \ldots, u_{n-1}) \neq 0$ for a certain j. Set $\hat{f} = f(u_1, \ldots, u_{n-1}, x_n)$. Then $\hat{f}$ is not the zero polynomial. But it vanishes on T_n. Hence it is a multiple of f_n. In particular, $r \geq t_n$. Now set $\tilde{f} = f - g_r x_n^{r-t_n} f_n$. Then $\tilde{f} \in I(T)$, but $\tilde{f} = h_0 + \cdots + h_{r-1} x_n^{r-1}$, with $h_i \in k[x_1, \ldots, x_{n-1}]$. So again we obtain a contradiction.

The conclusion is that $I(T) = \langle f_1, \ldots, f_n \rangle$. □

Theorem 1.2.23 (Alon's non-vanishing theorem) *Let $p \in k[x_1, \ldots, x_n]$ be of degree*

$$\sum_{i=1}^{n}(t_i - 1).$$

Suppose that the coefficient of $x_1^{t_1-1} \cdots x_n^{t_n-1}$ in p is nonzero. Then there is $u = (u_1, \ldots, u_n) \in T$ with $p(u) \neq 0$.

PROOF. We use a degree compatible term ordering, i.e., if $|\alpha| < |\beta|$ then $x^\alpha < x^\beta$.

Suppose that p vanishes on all elements of T. Then $p \in I(T)$. So by the previous lemma we can write

$$p = u_{i_1} f_{i_1} + \cdots + u_{i_m} f_{i_m}$$

where the u_{i_k} are terms, and

$$\mathrm{LM}(p) = \mathrm{LM}(u_{i_1} f_{i_1}) > \mathrm{LM}(u_{i_2} f_{i_2}) > \cdots > \mathrm{LM}(u_{i_m} f_{i_m}).$$

In particular this means that the degree of every $\mathrm{LM}(u_{i_j} f_{i_j})$ is at most $\sum_{i=1}^{n}(t_i - 1)$ (which is the degree of $\mathrm{LM}(p)$).

Note that $\deg(f_i) = t_i$. Suppose that the monomial $x_1^{t_1-1} \cdots x_n^{t_n-1}$ occurs in a $u_{i_k} f_{i_k}$. Then the degree of $u_{i_k} f_{i_k}$ has to exceed $\sum_{i=1}^{n}(t_i - 1)$, which is excluded. It follows that $x_1^{t_1-1} \cdots x_n^{t_n-1}$ occurs in no $u_{i_k} f_{i_k}$, which is a contradiction. □

Next we give an application in combinatorics of Alon's non-vanishing theorem.

Let p be a prime and $A, B \subset \mathbb{F}_p$. Then we define

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

Theorem 1.2.24 (Cauchy-Davenport) *Let A, B be non-empty subsets of $\mathbb{F}_p$. Then*

$$|A + B| \geq \min(p, |A| + |B| - 1).$$

PROOF. First assume that $|A| + |B| \leq p + 1$. Suppose that there exists a set $C \subset \mathbb{F}_p$ with $|C| = |A| + |B| - 2$, and $A + B \subset C$. (In other words, suppose that $|A + B| < |A| + |B| - 1$). Put

$$f = \prod_{c \in C}(x + y - c) \in \mathbb{F}_p[x, y].$$

Set $T_1 = A$ and $T_2 = B$. Then f vanishes on $T = T_1 \times T_2$. Also $\deg(f) = |C| = t_1 - 1 + t_2 - 1$. Furthermore, the coefficient of $x^{t_1-1} y^{t_2-1}$ is

$$\binom{t_1 - 1 + t_2 - 1}{t_1 - 1}.$$

But this is nonzero in $\mathbb{F}_p$ as $t_1 - 1 + t_2 - 1 = |A| + |B| - 2 \leq p - 1 < p$. So we have obtained a contradiction with Alon's theorem.

If $|A| + |B| > p + 1$, then we take subsets $A' \subset A$, $B' \subset B$ such that $|A'| + |B'| = p + 1$. Then

$$|A + B| \geq |A' + B'| \geq p.$$

(So $|A + B| = p$.) □

1.2.4 An application in cryptography: Polly-cracker

Here we briefly describe a cryptosystem based on the fact that it is difficult in general to compute Gröbner bases.

Alice wants to receive secret messages from Bob. To this end, Alice chooses a finite field $\mathbb{F}_q$ (for example $\mathbb{F}_2$) and works in $R = \mathbb{F}_q[x_1, \ldots, x_n]$

Alice chooses a $y \in \mathbb{F}_q^n$ and some polynomials $f_1, \ldots, f_s \in R$ with $f_i(y) = 0$. The public key is $f_1, \ldots, f_s$ and the secret key is y.

In order to send a message Bob first encodes it as an $m \in \mathbb{F}_q$, and then chooses polynomials $h_1, \ldots, h_s \in R$. Finally he sends the polynomial $f = m + \sum h_i f_i$ to Alice.

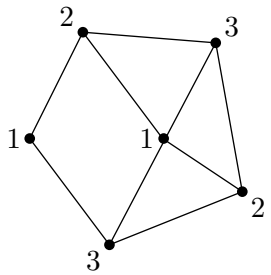
Alice can read the message, because $f(y) = m + \sum h_i f_i(y) = m$.

With Gröbner bases one can break the system. If G is a Gröbner basis of $\langle f_1, \ldots, f_s \rangle$ then $\bar{f}^G = m$.

So Alice needs to choose the polynomials f_i in such a way that a Gröbner basis of the ideal $\langle f_1, \ldots, f_s \rangle$ is very hard to compute.

A strategy to choose the f_i is the following: One takes a problem that is known to be very hard to solve (a so-called NP-complete problem, for example), and one reformulates a particularly difficult instance of that problem in terms of polynomial equations. Then one can realistically hope that a Gröbner basis is hard to compute.

For example, we can consider the 3-colouring problem from graph theory. A graph is given as $\Gamma = (V, E)$ (V: vertices, E: edges). A 3-colouring is a function $\varphi\colon V \to \{1, 2, 3\}$ with the property that if $(v, w) \in E$ then $\varphi(v) \neq \varphi(w)$. For example:



Now finding a 3-colouring is in general a very difficult problem (it is NP-complete).

Now we take $\mathbb{F}_2$ as the base field, and the indeterminates $t_{v,i}$ for $v \in V$ and $i \in \{1, 2, 3\}$. Let $B = B_1 \cup B_2 \cup B_3$ where

$$B_1 = \{t_{v,1} + t_{v,2} + t_{v,3} - 1 \mid v \in V\}$$
$$B_2 = \{t_{v,i} t_{v,j} \mid v \in V, 1 \leq i < j \leq 3\}$$
$$B_3 = \{t_{u,i} t_{v,i} \mid (u, v) \in E, 1 \leq i \leq 3\}.$$

Set $t_{v,i} = 1$ if the vertex v has colour i, and $t_{v,i} = 0$ otherwise. This defines a zero of all polynomials in B. Conversely, if we have a zero of all elements of B, then we can find a 3-colouring. Indeed, from the vanishing of the elements of $B_1 \cup B_2$ we find the colour of every vertex; and the vanishing of the elements of B_3 expresses that two adjacent vertices cannot have the same colour.

Experiments show that it is difficult to break the Polly Cracker system with Gröbner bases. However, there are other methods to try and do that, that make the system insecure. For example, observe that if $f = m + \sum_i h_i f_i$ is the message, then

$$m = f(0) - \sum_i h_i(0) f_i(0).$$

So it is enough to know the constant terms of the h_i in order to reconstruct the message. Sometimes it is possible to get the constant terms by studying how it is possible to obtain the polynomial f from the f_i.

1.2.5 Solving Sudoku

A *Sudoku* is a puzzle in which one has to place the numbers $1, \ldots, 9$ in an 9×9 grid according to certain rules. To go into detail, consider a 9×9-grid consisting of 81 empty boxes. This grid is itself divided into nine 3×3-blocks. Some of the boxes already have a number filled in. Then one has to complete the grid according to the following rule:

- In each row and in each column and in each 3×3-block each number $1, \ldots, 9$ has to appear exactly once.

The sudoku is said to be *well-posed* if there is exactly one solution. The key observation is that a sudoku is an instance of a *graph colouring problem*. There are diverse methods to code such a problem into polynomial equations. Here we follow the exposition in a recent book of Decker and Pfister (A First Course in Computational Algebraic Geometry, Cambridge University Press 2013).

3	5		9					8
							2	
			7		1	3		4
	3			4			8	2
1		5				9		7
2	8			7			3	
5		6	3		2			
	1							
9					6		7	5

In order to get polynomials from a sudoku we first number the boxes from 1 to 81. We work in the polynomial ring $R = \mathbb{Q}[x_1, \ldots, x_{81}]$. The value in each box of a completed sudoku is a zero of the polynomial $F_i(x_i) = \prod_{k=1}^{9}(x_i - k)$. We can write $F_i - F_j = h(x_i - x_j) + r$ (division with remainder) where r is a polynomial in the indeterminate x_j. Substituting x_j for x_i shows that $r = 0$, and we see that $F_i - F_j$ is divisible by $x_i - x_j$. For $i \neq j$ we define

$$G_{ij} = \frac{F_i - F_j}{x_i - x_j}.$$

Now we let E be the set of all pairs (i,j) such that $i<j$, and box i and box j are in the same row, column, or 3×3-block. So $(i,j)\in E$ if box i and box j cannot have the same colour. (E is the set of edges in the corresponding graph colouring problem.) Now we let I be the ideal of R generated by all F_i, $1\leq i\leq 81$, and G_{ij} for $(i,j)\in E$ (in total these are 891 polynomials).

Lemma 1.2.25 *Let $a=(a_1,\ldots,a_{81})\in\mathbb{C}^{81}$. Then $a\in V(I)$ if and only if $a_i\in\{1,\ldots,9\}$, and $a_i\neq a_j$ for $(i,j)\in E$.*

Proof. If $a_i\in\{1,\ldots,9\}$ for all i then $F_i(a)=0$ for all i. But then also $0=F_i(a)-F_j(a)=(a_i-a_j)G_{ij}(a)$. So if $a_i\neq a_j$ then $G_{ij}(a)=0$.

Conversely, suppose $a\in V(I)$. Then $F_i(a)=0$ so that $a_i\in\{1,\ldots,9\}$. Suppose that there is an $(i,j)\in E$ with $a_i=a_j=b$. Note that $F_i=(x_i-x_j)G_{ij}+F_j$. In this we substitute b for x_j, and get $F_i(x_i)=(x_i-b)G_{ij}(x_i,b)$. But since G_{ij} vanishes on a, we have $G_{ij}(b,b)=0$, and b is a zero of F_i with multiplicity at least 2. But that is impossible. □

Let S be a sudoku. Let $A\subset\{1,\ldots,81\}$ be such that for $i\in A$ the i-th box has the already filled in value a_i. Let I_S be the ideal of R generated by I along with x_i-a_i for $i\in A$.

Lemma 1.2.26 *Suppose that S is well-posed, and let $a=(a_1,\ldots,a_{81})$ be its unique solution. Let G be a Gröbner basis of I_S (with respect to any monomial ordering). Then $\overline{x}_i^G\stackrel{\circ}{=}a_i$.*

Proof. Set

$$J=\langle x_1-a_1,\ldots,x_{81}-a_{81}\rangle.$$

Then $f(a)=0$ for all $f\in J$. Conversely, suppose that $f(a)=0$. Perform division with remainder, and get $h_i\in R$ with $f=h_1(x_1-a_1)+\cdots+h_{81}(x_{81}-a_{81})+r$, and no monomial in r is divisible by the leading monomial of a x_i-a_i. But $\mathrm{LM}(x_i-a_i)=x_i$, and therefore r is constant. Evaluating in a we see that $r=0$. The conclusion is that J is precisely the ideal of all $f\in R$ with $f(a)=0$.

Because of Lemma 1.2.25, $V(I_S)=\{a\}$. So by Hilbert's Nullstellensatz (Theorem 1.2.17), $\sqrt{I_S}=\langle x_1-a_1,\ldots,x_{81}-a_{81}\rangle$. So for each i, I_S contains $(x_i-a_i)^{m_i}$, for some m_i. But it also contains the greatest common divisor of the univariate polynomials F_i and $(x_i-a_i)^{m_i}$, which is x_i-a_i. (Note that the greatest common divisor of two univariate polynomials p,q can be written as $ap+bq$, for some polynomials a,b.) So $I_S=\sqrt{I_S}=J$. Hence $\overline{x_i-a_i}^G=0$, whence $\overline{x}_i^G\stackrel{\circ}{=}a_i$. □

Of course, this is a very bad method for solving a sudoku! The polynomials corresponding to the sudoku above were tried in Magma; after more than seven hours on a 3.16 GHz processor, the system got out of memory (it needed more than 32GB).

1.3 Exercises

1. Consider the order $<_{\mathrm{rlex}}$ defined by $x^\alpha<_{\mathrm{rlex}}x^\beta$ if $\alpha_k<\beta_k$ where k is maximal with $\alpha_k\neq\beta_k$. Prove that $<_{\mathrm{rlex}}$ is a monomial order.

2. We use $<_{\mathrm{glex}}$ (graded lexicographical). Let $f_1=y^2-z$, $f_2=z^3-y$, $f_2=z^2-1$, and $g=xy^2z^2+xy-yz$. Find all possible remainders of g modulo f_1,f_2,f_3.

3. Compute $S(f,g)$ for $f = 4x^2z - 7y^2$, $g = xyz^2 + 3xz^4$, and $f = x^4y - z^2$, $g = 3xz^2 - y$, using the order $<_{\text{lex}}$.

4. Let $g_1 = x - z^2, g_2 = y - z^3 \in k[x,y,z]$. Prove that $G = \{g_1, g_2\}$ is a Gröbner basis with respect to $<_{\text{lex}}$. Prove that it is not a Gröbner basis with respect to $<_{\text{glex}}$. Compute a Gröbner basis of the ideal generated by G with respect to $<_{\text{glex}}$.

5. Let $g_1 = x^2y - 1, g_2 = xy^2 - x$. Compute a Gröbner basis of the ideal $I \subset k[x,y]$ generated by g_1, g_2, with respect to $<_{\text{glex}}$.

6. Let $I \subset k[x,y,z]$ be the ideal generated by $g_1 = x^2yz - yz - x$, $g_2 = xy^2z - xy - y$, $g_3 = xyz^2 - xy - z$. We use the order $<_{\text{glex}}$.

   (a) Compute $g_4 = S(g_1, g_2)$ (reduced modulo g_1, g_2, g_3), and $g_5 = S(g_1, g_3)$ (reduced modulo $g_1, \ldots, g_4$).

   (b) Compute $S(g_2, g_5)$, reduced modulo $g_1, \ldots, g_5$.

   (c) Prove that I is also generated by $h_1 = x^2z^2 - z^2 - x$, $h_2 = xz^3 - xz - z$, $h_3 = y - z$.

   (d) Compute $h_4 = S(h_1, h_2)$, $h_5 = S(h_1, h_4)$, $h_6 = S(h_2, h_5)$.

   (e) Prove that I is also generated by h_2, h_3, h_5, h_6. Show that they form a Gröbner basis of I.

   (f) Find $u_1, u_2, u_3 \in k[x,y,z]$ with $y - z = u_1g_1 + u_2g_2 + u_3g_3$

7. Let $g_1 = x^2 + 2y^2 - 3$, $g_2 = xy - y^2 + 3$ in $\mathbb{C}[x,y]$. Let I be the ideal generated by g_1, g_2.

   (a) Find a Gröbner basis of I with respect to $<_{\text{lex}}$.

   (b) Find a Gröbner basis of $I \cap \mathbb{C}[y]$.

   (c) Find all solutions of the equations $g_1 = g_2 = 0$.

8. Let $g_1 = x^2 + y^2 + z^2 - 4$, $g_2 = x^2 + 2y^2 - 5$, $g_3 = xz - 1$ in $\mathbb{C}[x,y,z]$. Let $I = \langle g_1, g_2, g_3 \rangle$.

   (a) We use the monomial order $<_{\text{lex}}$. Compute $g_4 = S(g_1, g_2)$ (reduced modulo g_1, g_2, g_3), and $g_5 = S(g_1, g_3)$ (reduced modulo $g_1, \ldots, g_4$), and $g_6 = S(g_3, g_5)$ (reduced modulo $g_1, \ldots, g_5$).

   (b) Prove that g_4, g_5, g_6 generate I, and that they form a Gröbner basis of I.

   (c) Compute a Gröbner basis of $I \cap \mathbb{C}[y,z]$ and of $I \cap \mathbb{C}[z]$.

   (d) Find all solutions of $g_1 = g_2 = g_3 = 0$.

9. Let $I \subset \mathbb{C}[x,y,z]$ be the ideal generated by $g_1 = x^2yz - yz - x$, $g_2 = xy^2z - xy - y$, $g_3 = xyz^2 - xy - z$. Let $f_1 = x - z^4 + z^2$, $f_2 = y - z$, $f_3 = z^7 - 2z^5 + z^3 - z$.

   (a) Prove that $I = \langle f_1, f_2, f_3 \rangle$ (hint: use the result of exercise 6).

   (b) Show that $\{f_1, f_2, f_3\}$ is a Gröbner basis of I with respect to the order $<_{\text{lex}}$ (with $x >_{\text{lex}} y >_{\text{lex}} z$).

   (c) Prove that f_3 is square free (one can compute $\gcd(f_3, f_3')$).

   (d) How many points $a = (a_1, a_2, a_3) \in \mathbb{C}^3$ are there with $f_1(a) = f_2(a) = f_3(a) = 0$?

10. A monomial order $<$ is said to be an *l-elimination order* if a monomial having at least one of $x_1, \ldots, x_l$ with positive exponent is bigger than every monomial in $k[x_{l+1}, \ldots, x_n]$.

    (a) Let $<_l$ be the order defined by $x^\alpha <_l x^\beta$ if $\alpha_1 + \cdots + \alpha_l < \beta_1 + \cdots + \beta_l$ or, if these are equal, if $\alpha <_{\text{glex}} \beta$. Show that $<_l$ is a monomial order and that it is an l-elimination order.

    (b) Let $<$ be an l-elimination order, and let G be a Gröbner basis of the ideal $I \subset k[x_1, \ldots, x_n]$, with respect to $<_l$. Prove that $G \cap k[x_{l+1}, \ldots, x_n]$ is a Gröbner basis of $I \cap k[x_{l+1}, \ldots, x_n]$.

11. Let $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$. Let $J \subset k[t_1, \ldots, t_r]$ the ideal consisting of all $g \in k[t_1, \ldots, t_r]$ with $g(f_1, \ldots, f_r) = 0$ (in other words, J is the ideal of all polynomial relations among the f_i). We want to compute generators of J.

    (a) Let $I \subset k[x_1, \ldots, x_n, t_1, \ldots, t_r]$ the ideal generated by $t_1 - f_1, \ldots, t_r - f_r$. Prove that
    $$J = I \cap k[t_1, \ldots, t_r]$$
    (hint for one inclusion: use the monomial order $<_{\text{lex}}$ with $t_i >_{\text{lex}} x_j$ for all i, j; let $h \in k[t_1, \ldots, t_r]$ and perform division with remainder to write $h = \sum g_i(t_i - f_i) + p$; show that $p \in k[x_1, \ldots, x_n]$, and in fact $p = h(f_1, \ldots, f_r)$).

    (b) Describe an algorithm for finding generators of J.

    (c) Let $r = 2$, $n = 1$, $x_1 = x$ and $f_1 = x^2$, $f_2 = x^3$. Find generators of J.

12. Let $I = \langle x^2, y^2 \rangle \subset k[x, y]$. Prove that $x + y \in \sqrt{I}$, by computing a Gröbner basis.

13. (Rational parametrisation.) Let
    $$C = \{\left(\frac{2t}{t^2+1}, \frac{1-t^2}{t^2+1}\right) \mid t \in \mathbb{C}, t^2 \neq -1\}.$$
    Let $f_1 = (t^2+1)y_1 - 2t$, $f_2 = (t^2+1)y_2 - 1 + t^2 \in \mathbb{C}[t, y_1, y_2]$, and set
    $$\Gamma = \{(s, u_1, u_2) \in \mathbb{C}^3 \mid f_1(s, u_1, u_2) = f_2(s, u_1, u_2) = 0\}.$$

    (a) Let $\pi_1 : \mathbb{C}^3 \to \mathbb{C}^2$ be defined by $\pi_1(s, u_1, u_2) = (u_1, u_2)$. Prove that $\pi_1(\Gamma) = C$.

    (b) Use the order $<_{\text{lex}}$, with $t >_{\text{lex}} y_1 >_{\text{lex}} y_2$. Compute $S(f_1, f_2)$ (reduced modulo f_1, f_2) and obtain f_3. In the same way compute $S(f_2, f_3)$ to obtain f_4 and $S(f_3, f_4)$ to obtain f_5.

    (c) Prove that $\{f_3, f_4, f_5\}$ is a Gröbner basis of the ideal generated by f_1, f_2, with respect to $<_{\text{lex}}$.

    (d) Find an ideal $J \subset \mathbb{C}[y_1, y_2]$ such that $V(J)$ is the closure of C.

14. Consider
    $$S = \{(uv, uv^2, u^2) \in \mathbb{C}^3 \mid u, v \in \mathbb{C}\}.$$
    A Gröbner basis of $\langle uv - x, uv^2 - y, u^2 - z \rangle$, with respect to the order $<_{\text{lex}}$ with $u >_{\text{lex}} v >_{\text{lex}} x >_{\text{lex}} y >_{\text{lex}} z$ is
    $$\{u^2 - z, uv - x, ux - vz, uy - x^2, v^2z - x^2, vx - y, vyz - x^3, x^4 - y^2z\}.$$
    Find an ideal $I \subset \mathbb{C}[x, y, z]$ such that the closure of S is $V(I)$. Find points on $V(I)$ that do not lie in S. (So $V(I)$ is strictly bigger than S.)

Chapter 2

Integer Factorisation

There are three types of algorithms that have to do with the problem of factorising an integer into its prime factors:

1) primality testing (probabilistic): such a test proves that n is not prime, or that n is prime with high probability (such tests are typically fast),

2) algorithms for proving that a given number n, which tests have shown to be probably prime, in fact is prime,

3) algorithms for factorising n, knowing that n is not prime.

The first algorithm that comes to mind, to solve all problems, is to divide n by all integers up to $\sqrt{n}$. For large n this becomes impossibly slow. However, it can be used to eliminate small factors ($\leq 10^5$ for example).

In this chapter we treat some other methods for these three problems. In the first section we describe the Miller-Rabin test, which is one of the most efficient primality tests. The second section has several methods for the last problem, i.e., we have an integer n and we know that it is not prime. The problem is to factorise it. The main algorithms that we describe are CFRAC (based on continued fractions) and ECM (the elliptic curve method). Finally in the last section we briefly touch upon a method for proving that a given number, which has a high probability of being prime, in fact is prime. This method uses elliptic curves, and is often denoted by the acronym ECPP (elliptic curve primality proving).

2.1 The Miller-Rabin primality test

The Miller-Rabin primality test was invented by Gary Miller

who in 1976 gave a deterministic version of the test (whose correctness depends on the Generalised Riemann Hypothesis), and Michael Rabin

who in the 1980's developed the probabilistic version of the test, which is still in use today.

Here we describe the test, following the article by René Schoof: *Four primality testing algorithms*, Algorithmic number theory; MSRI Publications 44, Cambridge University Press, Cambridge 2008, 101–126.

Lemma 2.1.1 *Let p be an odd prime and $e \geq 1$, x integers. Then $x^2 = 1 \bmod p^e$ implies $x = \pm 1 \bmod p^e$.*

PROOF. From $x^2 = 1 \bmod p^e$ we get that p^e divides $(x+1)(x-1)$. But p cannot divide both factors as p is odd. Hence p^e divides one of them; whence the statement. □

Lemma 2.1.2 *Let C_n be the cyclic group of order n, with generator g (so the order of g is n). Let $e \geq 1$ and $d = \gcd(e, n)$. There are exactly d elements $h \in C_n$ with $h^e = 1$.*

PROOF. Write $h = g^j$, where $1 \leq j \leq n$. Then $h^e = 1$ if and only if $g^{ej} = 1$, which is equivalent to $n|ej$. In turn this is the same as saying $\frac{n}{d}|\frac{e}{d}j$. But the latter happens if and only if $\frac{n}{d}|j$. So for j we have the possibilities $j = \frac{n}{d}, 2\frac{n}{d}, \ldots, d\frac{n}{d}$. □

Theorem 2.1.3 *Let $n > 9$ be an integer that is composite and odd. Write $n - 1 = 2^k m$, where $k \geq 1$ and m is odd. Set*

$$B = \{x \in (\mathbb{Z}/n\mathbb{Z})^* \mid x^m = 1 \text{ or there is } i \text{ with } 0 \leq i < k \text{ and } x^{m2^i} = -1\}.$$

Then

$$\frac{|B|}{\varphi(n)} \leq \frac{1}{4}$$

(where $\varphi(n) = |(\mathbb{Z}/n\mathbb{Z})^*|$).

PROOF. Let $l \in \mathbb{Z}$ be maximal such that 2^l divides $p - 1$ for all prime factors p of n. Set $B' = \{x \in (\mathbb{Z}/n\mathbb{Z})^* \mid x^{m2^{l-1}} = \pm 1\}$. We claim that $B \subset B'$. To see this, let $x \in B$. If $x^m = 1$ then $x^{m2^{l-1}} = 1$. Secondly, suppose that $x^{m2^i} = -1$ for an i with $0 \leq i < k$. By x we also denote the integer congruent to $x \bmod n$, and $1 < x < n$. Then $x^{m2^{i+1}} = 1 \bmod p$ for all primes p that divide n. So for all such p, the exact power of 2 that divides the order of $x \bmod p$ is 2^{i+1}. So $2^{i+1}|p - 1$ (as $(\mathbb{Z}/p\mathbb{Z})^*$ has $p - 1$ elements). It follows that $i < l$. Furthermore, if $i = l - 1$ then $x^{m2^{l-1}} = -1$ and if $i < l - 1$ then $x^{m2^{l-1}} = 1$. The claim is proved.

Now write $n = \prod_{i=1}^{s} p_i^{e_i}$, where the p_i are distinct (odd) primes. Then by the chinese remainder theorem we get

$$(\mathbb{Z}/n\mathbb{Z})^* = (\mathbb{Z}/p_1^{e_1}\mathbb{Z})^* \times \cdots \times (\mathbb{Z}/p_s^{e_s}\mathbb{Z})^*.$$

Now the group $(\mathbb{Z}/p_i^{e_i}\mathbb{Z})^*$ is cyclic of order $(p_i-1)p_i^{e_i-1}$. So by Lemma 2.1.2 the number of $y \in (\mathbb{Z}/p_i^{e_i}\mathbb{Z})^*$ with $y^{m2^{l-1}} = 1$ is $\gcd((p_i-1)p_i^{e_i-1}, m2^{l-1}) = \gcd(p_i-1, m)2^{l-1}$ (by the definition of l we see that $2^{l-1} | p_i - 1$; furthermore, $p_i \not| m$). For this reason, the number of $x \in (\mathbb{Z}/n\mathbb{Z})^*$ with $x^{m2^{l-1}} = 1$ is $\prod_{i=1}^{s} \gcd(p_i-1, m)2^{l-1}$. Analogously, the number of solutions in $(\mathbb{Z}/p_i^{e_i}\mathbb{Z})^*$ of the equation $X^{m2^l} = 1$ is $\gcd(p_i-1, m)2^l$. This is twice the number of solutions to the equation $X^{m2^{l-1}} = 1$. So by Lemma 2.1.1, it follows that the number of solutions to the equation $X^{m2^{l-1}} = -1$ is the same as the number of solutions to $X^{m2^{l-1}} = 1$. We conclude that

$$|B'| = 2\prod_{i=1}^{s} \gcd(p_i-1, m)2^{l-1}.$$

Now suppose that $|B|/\varphi(n) > \frac{1}{4}$. Since $B \subset B'$ and $\varphi(n) = \prod_{i=1}^{s}(p_i-1)p_i^{e_i-1}$ we get

$$\frac{1}{4} < 2\prod_{i=1}^{s} \frac{\gcd(p_i-1, m)2^{l-1}}{(p_i-1)p_i^{e_i-1}}. \tag{2.1}$$

Observe that $\gcd(p_i-1, m)2^{l-1}$ divides $\frac{p_i-1}{2}$. From this it follows that the right hand side of (2.1) is smaller than $2^{1-s}\prod_{i=1}^{s} \frac{1}{p_i^{e_i-1}}$. So $s \leq 2$. Suppose first that $s = 2$. Then also $e_1 = e_2 = 1$, so that $n = pq$, where p, q are distinct primes. Then (2.1) is equivalent to

$$2 > \frac{p-1}{\gcd(p-1, m)2^l} \frac{q-1}{\gcd(q-1, m)2^l}.$$

Both factors on the right are positive integers; so both have to be 1. Hence $p - 1 = \gcd(p-1, m)2^l$ and $q - 1 = \gcd(q-1, m)2^l$. Write $p - 1 = a_p 2^l$, $q - 1 = a_q 2^l$ with a_p, a_q odd. Then a_p, a_q divide m. Consider the equation $pq = 1 + 2^k m$ modulo a_p. Since $p = 1 \bmod a_p$ we get $a_p | q - 1$. Analogously, $a_q | p - 1$. This implies that $a_p = a_q$, which is a contradiction.

It follows that $s = 1$ and $n = p^e$, $e \geq 2$. Then (2.1) is equivalent to

$$4 > \frac{1}{2}\frac{(p-1)p^{e-1}}{\gcd(p-1, m)2^{l-1}} = \frac{p-1}{\gcd(p-1, m)2^l} p^{e-1} \geq p^{e-1}$$

(as $\gcd(p-1, m)2^l$ divides $p-1$). So the only possibility is $p = 3$, $e = 2$, but that contradicts $n > 9$. □

Now the main idea of the next algorithm is as follows. If n is prime then B is all of $(\mathbb{Z}/n\mathbb{Z})^*$ (proof: exercise!). Therefore, if a random $x \in (\mathbb{Z}/n\mathbb{Z})^*$ does not lie in B, then that proves that n is not prime. On the other hand, if random elements x continue to lie in B, then probably n is prime, as otherwise the proportion of elements in B is less than a quarter.

Algorithm 2.1.4 *Given:* $n > 9$ *odd.*
We return: FALSE *(meaning that* n *certainly is not prime) or* FAIL *(meaning that* n *is likely to be prime).*

1. *Take a random integer x with $1 < x < n$ (uniform distribution).*
2. *If $\gcd(x, n) > 1$ return* FALSE.
3. *If $x^{n-1} \neq 1 \bmod n$ then return* FALSE.
4. *Write $n - 1 = 2^k m$, with m odd. Compute the minimal $i \geq 0$ with $x^{m2^i} = 1 \bmod n$. If $i > 0$ and $x^{m2^{i-1}} \neq -1 \bmod n$ then return* FALSE. *Otherwise return* FAIL.

Note that the i in Step 4 exists as $x^{m2^k} = 1 \bmod n$ by Step 3.

Theorem 2.1.5 *If the previous algorithm returns* FALSE *then n is composite. Furthermore, the probability of n being composite and the algorithm returning* FAIL *is $\leq \frac{1}{4}$.*

PROOF. Suppose that $i > 0$ and $x^{m2^i} = 1 \bmod n$. If n is prime, then by Lemma 2.1.1 we see that $x^{m2^{i-1}} = \pm 1 \bmod n$. But it cannot be 1 as i is minimal. If it is not -1 either, then we conclude that n cannot be prime. If FAIL is returned then either $x^m = 1 \bmod n$ or $x^{m2^{i-1}} = -1 \bmod n$, implying that $x \in B$ (the set from Theorem 2.1.3). (NB: after Step 2 we know that $x \bmod n$ lies in $(\mathbb{Z}/n\mathbb{Z})^*$.) But the probability of a random element lying in B is $\leq \frac{1}{4}$. □

Example 2.1.6 Let $n = 1729$. This is a Carmichael number, i.e., it is not prime, but $x^{n-1} = 1 \bmod n$ for all positive integers $x < n$, with $\gcd(x, n) = 1$. We have $n - 1 = 2^6 \cdot 27$. So we take $m = 27$. Let $x = 3$. Then $x^{27} = 664 \bmod n$ and $x^{27 \cdot 2} = 1 \bmod n$. Hence the Miller-Rabin test shows that n is not prime.

2.2 Factorisation

2.2.1 The method of Fermat

Note that we may assume that n is odd. Set

$$A = \{(a, b) \mid 0 < a \leq b \text{ and } n = ab\}$$

and

$$B = \{(s, t) \mid 0 \leq s < t \text{ and } n = t^2 - s^2\}.$$

There is a bijection $\sigma \colon A \to B$ with $\sigma(a, b) = \left(\frac{b-a}{2}, \frac{b+a}{2}\right)$.

Indeed:

$$\left(\frac{b+a}{2}\right)^2 - \left(\frac{b-a}{2}\right)^2 = ab = n$$

and

$$\sigma^{-1}(s, t) = (t - s, t + s).$$

The conclusion is that finding a factorisation $n = ab$ is equivalent to finding s and t with $n = t^2 - s^2$. So we have the following algorithm.

Algorithm 2.2.1 *Given: n odd and non-prime.*
We compute: $a, b > 1$ with $n = ab$.

1. *Set* $t_0 = \lceil \sqrt{n} \rceil = \min\{k \in \mathbb{Z} \mid k \geq \sqrt{n}\}$.
2. *If* $t_i^2 - n = s_i^2$ *is a square then set* $a = (t_i + s_i)$, *and* $b = (t_i - s_i)$. *Otherwise set* $t_{i+1} = t_i + 1$, *and continue.*

This algorithm terminates correctly because if $n = t^2 - s^2$ with $s < t$ then $t^2 = n + s^2$ implies $t > \sqrt{n}$.

Example 2.2.2 Take $n = 39$. then

$$\begin{array}{rcl} t_0 = 7 & \Rightarrow & 7^2 - 39 = 10 \\ t_1 = 8 & \Rightarrow & 8^2 - 39 = 25 = 5^2 \end{array}$$

hence $(8 + 5)(8 - 5) = 39$.

This algorithm works well if s is small, because in that case t is close to $\sqrt{n}$ and hence only a few steps are needed, even if the factors themselves are big. So if one wants to construct a number that is difficult to factor (as in the cryptosystem RSA, for example) then the factors cannot be chosen close to each other.

Pierre de Fermat (1601 - 1665)

was a french lawyer, in mathematics most famous for his work in number theory, although he did not publish most of his results, preferring to mention the problems he had solved to other mathematicians and challenging them to also come up with a solution.

2.2.2 The continued fraction method (CFRAC)

Some more advanced methods use similars idea to Fermat's algorithm. To illustrate the main point let's try to factorise $n = 1649$ with Fermat's method. The first three steps are

t	$t^2 - n$
41	$32 = 2^5$
42	$115 = 5 \cdot 23$
43	$200 = 2^3 \cdot 5^2$

We see that we get no square yet. However, the product of the first and third entries in the second column *is* a square: $32 \cdot 200 = (2^4 \cdot 5)^2$. Also note that $(t_1^2 - n)(t_2^2 - n) \equiv (t_1 t_2)^2 \bmod n$. Therefore we have that $(41 \cdot 43)^2 \equiv 80^2 \bmod n$. Using that $41 \cdot 43 \equiv 114 \bmod n$ we see that n divides $114^2 - 80^2 = (114 - 80)(114 + 80)$. But n does not divide any of the two factors on the right. Therefore $\gcd(n, 114 - 80)$ and $\gcd(n, 114 + 80)$ are nontrivial factors of n. In fact, $\gcd(n, 114 - 80) = 17$, $\gcd(n, 114 + 80) = 97$ and $n = 17 \cdot 97$.

We now try to base a more general method on this example. Instead of looking for s and t with $n = t^2 - s^2$ we try to find x and y with

- $x^2 - y^2 = 0 \bmod n$;

- $0 < y < x < n$;

- $x + y \neq n$.

If we have such x, y then we can factorise n because $(x+y)(x-y)$ is divisible by n, but $x+y$ and $x-y$ are not. Hence $a = \gcd(x+y, n)$ and $b = \gcd(x-y, n)$ are nontrivial factors of n (which means $a, b \neq 1$ and $a, b \neq n$).

So the idea is to find x with $0 < x < n$ such that $x^2 \bmod n$ is a square y^2. Not all x can be chosen for this: for example if $n = 15$ and $x = 2$, then $x^2 \bmod n = 4$ but $y = 2$ doesn't work as $y = x$. But also $x = 13$ is not good as $x^2 \bmod n = 4$ but with $y = 2$ we get $x + y = n$. (Note that in this case $x = -2 \bmod n$.) A better try is $x = 7$ with which we find $7^2 \bmod n = 4$ and

$$\gcd(7+2, 15) = 3 \qquad \text{and} \qquad \gcd(7-2, 15) = 5.$$

Notation. By $x \ \underline{\bmod}\ n$ we denote the unique integer with $-\frac{n}{2} < s \leq \frac{n}{2}$, congruent to $x \bmod n$.

Definition 2.2.3 *A set $B = \{p_0 = -1, p_1, \ldots, p_r\}$ with p_i prime for $i \geq 1$, is called a* factor base.

Definition 2.2.4 *An integer k is called a* B-number *(or* B-smooth*) if $k = p_0^{e_0} p_1^{e_1} \cdots p_r^{e_r}$.*

In order to find x such that $x^2 \bmod n$ is a square we try the following scheme:

- we choose a factor base;

- we generate a lot of integers b_i such that $b_i^2 \ \underline{\bmod}\ n$ is B-smooth (note that that was happening in the little example at the beginning of this section).

The aim is to find a product $b_{i_1} \cdots b_{i_m}$ such that

$$(b_{i_1} \cdots b_{i_m})^2 \bmod n = p_0^{e_0} p_1^{e_1} \cdots p_r^{e_r}$$

with e_i even for all i, and hence $p_1^{e_1} \cdots p_r^{e_r} = y^2$.

Example 2.2.5 Take $n = 377$ and $B = \{-1, 2\}$. Then

$$\begin{aligned} 19^2 &= -16 \ \underline{\bmod}\ n \text{ and } -16 = p_0 p_1^2 \\ 97^2 &= -16 \ \underline{\bmod}\ n \text{ and } -16 = p_0 p_1^2. \end{aligned}$$

Therefore $(19 \cdot 97)^2 = p_0^2 p_1^4 \bmod n$. So we set $x = 19 \cdot 97 = 335 \bmod n$, and $y = 16$. Indeed, $\gcd(335 + 16, 377) = 13$ and $\gcd(335 - 16, 377) = 29$ are factors of n.

So the idea is to combine the factorisations of several $b_i^2 \ \underline{\bmod}\ n$ such that a square arises. From our example above it may seem that we have a good way of doing this: generate a few of the numbers $t^2 - n$ like in Fermat's method, and use the values of t as the b_i. However, although this worked well in the example, in general those t will not have the property that $t^2 \ \underline{\bmod}\ n$ is a product of small primes. A famous method for generating b_i with good properties for this purpose is based on the classical theory of continued fractions.

Continued fractions

A continued fraction is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4}}}}$$

with $a_i \in \mathbb{Z}$, $a_0 \geq 0$ and $a_i \geq 1$ for $i \geq 1$.

Notation. The expression above is denoted by $[a_0; a_1, a_2, a_3, a_4]$.

More generally we say that a continued fraction is a sequence of the form $[a_0; a_1, \ldots, a_n]$, with $a_i \in \mathbb{Z}$, $a_0 \geq 0$, $a_i \geq 1$ for $i \geq 1$. This corresponds to a rational number γ, which is defined as follows. If $n = 0$ then $\gamma = a_0$. If $n > 0$, then let γ' be the rational number corresponding to $[a_1; a_2, \ldots, a_n]$, and set

$$\gamma = a_0 + \frac{1}{\gamma'}.$$

Let $x > 0$ in $\mathbb{R}$. We want to find $a_0, a_1, \ldots$ such that the continued fractions $[a_0; a_1, \ldots, a_k]$ are good approximations of x; we want them to converge to x when $k \to \infty$ (it is not clear yet that it is always possible to find such a_i; we will show that in this section).

The a_i, if they exist, must be defined in the following way: Write $x = a_0 + \frac{1}{x_1}$ with $a_0 = \lfloor x \rfloor = \max\{n \in \mathbb{Z} \mid n \leq x\}$.

Set $x_1 = \frac{1}{x - a_0}$. Next, write $x_1 = a_1 + \frac{1}{x_2}$ with $a_1 = \lfloor x_1 \rfloor$ and hence $x_2 = \frac{1}{x_1 - a_1}$ and so on.

More formally, set $x_0 = x$ and for $i \geq 0$:

$$a_i = \lfloor x_i \rfloor,$$
$$x_{i+1} = \frac{1}{x_i - a_i}.$$

Example 2.2.6 Let $x = \sqrt{2}$. Then $x_0 = x$ and

$$\begin{aligned} a_0 = 1 \quad &\Rightarrow \quad x_1 = \frac{1}{\sqrt{2} - 1} = 1 + \sqrt{2} \\ a_1 = 2 \quad &\Rightarrow \quad x_2 = \frac{1}{\sqrt{2} - 1} = 1 + \sqrt{2} \\ a_2 = 2 \quad &\Rightarrow \quad x_3 = \frac{1}{\sqrt{2} - 1} = 1 + \sqrt{2} \end{aligned}$$

continuing like this we find $a_i = 2$ for all $i \geq 1$. Now, for example,

$$[1; 2, 2, 2, 2] = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}}} = \frac{41}{29}.$$

We have that $2 - (\frac{41}{29})^2 = \frac{1}{841}$; so it is a reasonable approximation of $\sqrt{2}$.

Lemma 2.2.7 *For $x \in \mathbb{Q}$ the algorithm for computing the continued fraction corresponding to x terminates (that is, at a certain point we find $x_i - a_i = 0$). The resulting continued fraction is equal to x.*

Proof. Write $x = \frac{a}{b}$ and $a = q_1 b + r_1$ with $0 \leq r_1 < b$. Then

$$a_0 = q_1 \quad \text{and} \quad x_1 = \frac{1}{\frac{a}{b} - q_1} = \frac{b}{r_1}.$$

Now write $b = q_2 r_1 + r_2$ with $0 \leq r_2 < r_1$ and get

$$a_1 = q_2 \text{ and } x_2 = \frac{r_1}{r_2}.$$

So we find a sequence of r_i with $0 \leq r_{i+1} < r_i$; therefore the algorithm must terminate.

By induction on the number of steps one shows that the continued fraction that is found this way, is equal to x. □

When the continued fraction for x produces the integers $a_0, a_1, \ldots$, we write $\frac{b_i}{c_i} = [a_0; a_1, \ldots, a_i]$ where $\gcd(b_i, c_i) = 1$. These rational numbers are called the *convergents* of the continued fraction for x.

Example 2.2.8 For $x = \sqrt{2}$ we get

$$1 + \frac{1}{2 + \frac{1}{2}} = \frac{7}{5} = \frac{b_2}{c_2}.$$

Theorem 2.2.9 *Let $a_0, a_1, \ldots,$ be the integers that are produced by the algorithm for computing the continued fraction for $x \in \mathbb{R}$, $x > 0$. Set*

$$\begin{array}{lll} \beta_0 = a_0, & \beta_1 = a_0 a_1 + 1, & \beta_i = a_i \beta_{i-1} + \beta_{i-2}, \\ \gamma_0 = 1, & \gamma_1 = a_1, & \gamma_i = a_i \gamma_{i-1} + \gamma_{i-2}, \end{array}$$

for $i \geq 2$. Then

1) $\beta_i \gamma_{i-1} - \beta_{i-1} \gamma_i = (-1)^{i-1}$, $i \geq 1$;

2) $\gcd(\beta_i, \gamma_i) = 1$;

3) $b_i = \beta_i$ *and* $c_i = \gamma_i$, *where* $b_i, c_i \in \mathbb{Z}_{\geq 0}$ *are such that* $\gcd(b_i, c_i) = 1$ *and* $\frac{b_i}{c_i} = [a_0; a_1, \ldots, a_i]$.

Proof. 1) We use induction. For $i = 1$ we have

$$\beta_1 \gamma_0 - \beta_0 \gamma_1 = (a_0 a_1 + 1)1 - a_0 a_1 = 1 = (-1)^0.$$

For $i \geq 1$ we get, using the induction hypothesis

$$\begin{aligned} \beta_{i+1} \gamma_i - \beta_i \gamma_{i+1} &= (a_{i+1} \beta_i + \beta_{i-1}) \gamma_i - \beta_i (a_{i+1} \gamma_i + \gamma_{i-1}) \\ &= \beta_{i-1} \gamma_i - \beta_i \gamma_{i-1} \\ &= -(-1)^{i-1} \\ &= (-1)^i. \end{aligned}$$

2) If p divides γ_i and β_i then by 1) it also divides $(-1)^{i-1}$, which is impossible for a prime p.

3) Also here we use induction on i. For $i = 0$ we get

$$\frac{b_0}{c_0} = a_0 = \frac{\beta_0}{\gamma_0}$$

and for $i = 1$

$$\frac{b_1}{c_1} = a_0 + \frac{1}{a_1} = \frac{\beta_1}{\gamma_1}.$$

Note that $[a_0; a_1, \ldots, a_i, a_{i+1}]$ is obtained from $[a_0; a_1, \ldots, a_i]$ by substituting $a_i + \frac{1}{a_{i+1}}$ for a_i. By induction we now have

$$[a_0; a_1, \ldots, a_i] = \frac{\beta_i}{\gamma_i} = \frac{a_i\beta_{i-1} + \beta_{i-2}}{a_i\gamma_{i-1} + \gamma_{i-2}}$$

hence

$$\begin{aligned}
[a_0; a_1, \ldots, a_{i+1}] &= \frac{\left(a_i + \frac{1}{a_{i+1}}\right)\beta_{i-1} + \beta_{i-2}}{\left(a_i + \frac{1}{a_{i+1}}\right)\gamma_{i-1} + \gamma_{i-2}} \\
&= \frac{a_{i+1}(a_i\beta_{i-1} + \beta_{i-2}) + \beta_{i-1}}{a_{i+1}(a_i\gamma_{i-1} + \gamma_{i-2}) + \gamma_{i-1}} \\
&= \frac{a_{i+1}\beta_i + \beta_{i-1}}{a_{i+1}\gamma_i + \gamma_{i-1}} \\
&= \frac{\beta_{i+1}}{\gamma_{i+1}}.
\end{aligned}$$

It follows that $\beta_{i+1} = b_{i+1}$ and $\gamma_{i+1} = c_{i+1}$ (since $\gcd(\beta_{i+1}, \gamma_{i+1}) = 1$). □

Remark 2.2.10 It follows that $b_ic_{i-1} - c_ib_{i-1} = (-1)^{i-1}$. Divide by c_ic_{i-1} to get

$$\frac{b_i}{c_i} - \frac{b_{i-1}}{c_{i-1}} = \frac{(-1)^{i-1}}{c_ic_{i-1}}$$

and hence

$$\left|\frac{b_i}{c_i} - \frac{b_{i-1}}{c_{i-1}}\right| = \frac{1}{c_ic_{i-1}}.$$

But $c_i > c_{i-1} > \ldots \geq 1$ so

$$\left|\frac{b_i}{c_i} - \frac{b_{i-1}}{c_{i-1}}\right| \longrightarrow 0.$$

Theorem 2.2.11 *The notation is as in Theorem 2.2.9. Then x is always between two successive convergents. In particular, $\frac{b_i}{c_i}$ converges to x when $i \to \infty$.*

Proof. For $i \geq 0$ we set $\tilde{x}_i = x_i - a_i$. Then for $i \geq 1$ we claim that

$$x = \frac{b_i + \widetilde{x}_ib_{i-1}}{c_i + \widetilde{x}_ic_{i-1}}.$$

This is shown by induction. For $i = 1$ the expression on the right is

$$\frac{b_1 + \tilde{x}_1 b_0}{c_1 + \tilde{x}_1 c_0} = \frac{a_0 a_1 + 1 + (x_1 - a_1)a_0}{a_1 + (x_1 - a_1)} = \frac{a_0 x_1 + 1}{x_1} = a_0 + \frac{1}{x_1} = x.$$

Now suppose that the claim halds for a certain $i \geq 1$. Then

$$\frac{b_{i+1} + \tilde{x}_{i+1} b_i}{c_{i+1} + \tilde{x}_{i+1} c_i} = \frac{a_{i+1} b_i + b_{i-1} + (x_{i+1} - a_{i+1})b_i}{a_{i+1} c_i + c_{i-1} + (x_{i+1} - a_{i+1})c_i} = \frac{x_{i+1} b_i + b_{i-1}}{x_{i+1} c_i + c_{i-1}} = \frac{\frac{1}{x_i - a_i} b_i + b_{i-1}}{\frac{1}{x_i - a_i} c_i + c_{i-1}} = \frac{b_i + \tilde{x}_i b_{i-1}}{c_i + \tilde{x}_i c_{i-1}}.$$

But by induction the latter is equal to x.

Now $\frac{b_i + \widetilde{x}_i b_{i-1}}{c_i + \widetilde{x}_i c_{i-1}} < \frac{b_i}{c_i}$ if and only if $\widetilde{x}_i(b_i c_{i-1} - b_{i-1} c_i) > 0$. But by Theorem 2.2.9 that is the same as $(-1)^{i-1}\widetilde{x}_i > 0$, which in turn is equivalent to i being odd. In the same way it is shown that $\frac{b_i + \widetilde{x}_i b_{i-1}}{c_i + \widetilde{x}_i c_{i-1}} < \frac{b_{i-1}}{c_{i-1}}$ if and only if i is even. It follows that x is always between two successive convergents.

From Remark 2.2.10 we have that the distance between $\frac{b_i}{c_i}$ and $\frac{b_{i-1}}{c_{i-1}}$ goes to 0. Hence $\frac{b_i}{c_i}$ converges to a limit that has to be x (as x is always between the two). □

Factorisation with continued fractions

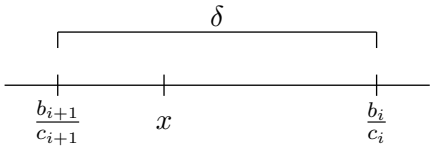
Lemma 2.2.12 *Let $x \in \mathbb{R}$, $x > 1$. Let $\frac{b_i}{c_i}$ be the convergents of the continued fraction for x. Then*

$$|b_i^2 - c_i^2 x^2| < 2x.$$

Proof. By Remark 2.2.10 we have

$$\left| \frac{b_{i+1}}{c_{i+1}} - \frac{b_i}{c_i} \right| = \frac{1}{c_{i+1} c_i}.$$

Set $\delta = \frac{1}{c_{i+1} c_i}$. Then by Theorem 2.2.11 we have the following picture.



Now

$$|b_i^2 - c_i^2 x^2| = |c_i^2 x^2 - b_i^2| = c_i^2 \left| x - \frac{b_i}{c_i} \right| \left| x + \frac{b_i}{c_i} \right|.$$

And $\left| x - \frac{b_i}{c_i} \right| < \delta$ while

$$\left| x + \frac{b_i}{c_i} \right| = x + \frac{b_i}{c_i} = 2x + \frac{b_i}{c_i} - x < 2x + \delta$$

hence

$$|b_i^2 - c_i^2 x^2| < c_i^2 \delta (2x + \delta).$$

Therefore

$$|b_i^2 - c_i^2 x^2| - 2x < 2x\left(-1 + \delta c_i^2 + \frac{\delta^2 c_i^2}{2x}\right) = 2x\left(-1 + \frac{c_i}{c_{i+1}} + \frac{1}{2xc_{i+1}^2}\right).$$

But $\frac{1}{2xc_{i+1}^2} < \frac{1}{c_{i+1}}$ because $x > 1$ and $c_{i+1} \geq 1$. Moreover, $c_i + 1 \leq c_{i+1}$ because $c_i < c_{i+1}$. Hence

$$|b_i^2 - c_i^2 x^2| - 2x < 2x\left(-1 + \frac{c_i}{c_{i+1}} + \frac{1}{c_{i+1}}\right) \leq 2x\left(-1 + \frac{c_{i+1}}{c_{i+1}}\right) = 0$$

and therefore

$$|b_i^2 - c_i^2 x^2| < 2x.$$

□

Corollary 2.2.13 *Let $n > 16$ be an integer and $\frac{b_i}{c_i}$ the convergents of the continued fraction of $\sqrt{n}$. Then $|b_i^2 \mod n| < 2\sqrt{n}$.*

Proof. By Lemma 2.2.12 we have $|b_i^2 - c_i^2 n| < 2\sqrt{n}$. But $b_i^2 - c_i^2 n$ is an integer congruent to $b_i^2 \bmod n$ and contained in the interval $(-2\sqrt{n}, 2\sqrt{n})$. Hence it is contained in the interval $\left(-\frac{n}{2}, \frac{n}{2}\right)$ (as $n > 16$). It follows that $b_i^2 - c_i^2 n$ is equal to $b_i^2 \mod n$. □

Idea of the factorisation algorithm: Let $B = \{p_0 = -1, p_1, \ldots, p_r\}$ be a factor base. Let $\frac{b_i}{c_i}$ be the convergents of the continued fraction of $\sqrt{n}$.

Then $b_i^2 \mod n$ is always "small". So among the $b_i^2 \mod n$ there are often B-smooth numbers. On the other hand, $b_i \bmod n$ "jumps" through the entire interval $[0, n]$. So one can reasonably hope to quickly find $b_{i_1}, \ldots, b_{i_t}$ such that $(b_{i_1} \cdots b_{i_t})^2 \bmod n$ is a square.

Algorithm 2.2.14 *Given: n which is not prime.*
Calculate: two factors of n.

1. *Choose a factor base $B = \{p_0 = -1, p_1, \ldots, p_r\}$.*
2. *Compute the continued fraction for $\sqrt{n}$ and find $a_0, a_1, a_2, \ldots$.*
3. *Compute $b_0 = a_0$, $b_1 = a_0 a_1 + 1$, $b_{i+1} = a_{i+1} b_i + b_{i-1}$.*
4. *For the b_i such that $b_i^2 \mod n$ is B-smooth we write $b_i^2 \mod n = p_0^{e_0^i} \cdots p_r^{e_r^i}$ until we find a product $(b_{i_1} \cdots b_{i_t})^2 \bmod n = p_0^{e_0} \cdots p_r^{e_r}$ with e_i even for all i. Set $x = b_{i_1} \cdots b_{i_t}$ and $y = p_0^{\frac{e_0}{2}} \cdots p_r^{\frac{e_r}{2}}$.*
5. *If we are not unlucky, then $\gcd(n, x + y)$ and $\gcd(n, x - y)$ are factors of n.*

Remark 2.2.15 The algorithm is based on heuristic ideas; it is not guaranteed that it manages to factor n. However, we know that n is composite. Hence if we do not find the factorisation in reasonable time we can try again; for example with a larger factor base. Implementations have shown that the algorithm works well in practice.

Example 2.2.16 (Factorisation of Fermat numbers)

The Fermat numbers are $F_n = 2^{2^n} + 1$.

Fermat thought that these were all prime numbers. In fact, F_n is prime for $0 \leq n \leq 4$. However, Euler has shown that

$$F_5 = 641 \cdot 6700417.$$

In 1880 Landry proved that

$$F_6 = 274177 \cdot 67280421310721.$$

And in 1970 Morison and Brillhart, using the continued fraction method on an early computer, got

$$F_7 = 59649589127497217 \cdot P_{22}$$

where P_{22} is a prime of 22 digits. (On my laptop this factorisation now takes less than a millisecond.)

Remark 2.2.17 In order to compute the continued fraction for $\sqrt{n}$ the following observation can help. For $\alpha \in \mathbb{R}$, $\alpha > 0$ and $m \in \mathbb{Z}$, $m > 0$, we have $\left\lfloor \frac{\alpha}{m} \right\rfloor = \left\lfloor \frac{\lfloor \alpha \rfloor}{m} \right\rfloor$.

Proof. Write $\alpha = u + \beta$ with u an integer and $\beta \in \mathbb{R}$, $0 \leq \beta < 1$. Write $u = qm + r$ with $0 \leq r < m$. Then

$$\left\lfloor \frac{\alpha}{m} \right\rfloor = \left\lfloor \frac{u}{m} + \frac{\beta}{m} \right\rfloor = \left\lfloor q + \frac{r+\beta}{m} \right\rfloor = q$$

because $r + \beta < m$ and hence $\frac{r+\beta}{m} < 1$. On the other hand,

$$\left\lfloor \frac{\lfloor \alpha \rfloor}{m} \right\rfloor = \left\lfloor \frac{u}{m} \right\rfloor = \left\lfloor q + \frac{r}{m} \right\rfloor = q.$$

□

Example 2.2.18 We want to factorise $n = 33$.

The continued fraction for $\sqrt{33}$:

$$\begin{array}{ll}
a_0 = 5 & \tilde{x}_0 = x - a_0 = \sqrt{33} - 5 \\
 & x_1 = \frac{1}{\sqrt{33}-5} = \frac{5+\sqrt{33}}{8} \\
a_1 = 1 & \tilde{x}_1 = x_0 - a_1 = \frac{-3+\sqrt{33}}{8} \\
 & x_2 = \frac{1}{\tilde{x}_1} = \frac{8}{-3+\sqrt{33}} = \frac{3+\sqrt{33}}{3} \\
a_2 = 2 & \tilde{x}_2 = x_1 - a_2 = \frac{-3+\sqrt{33}}{3} \\
 & x_3 = \frac{3+\sqrt{33}}{8} \\
a_3 = 1 & \tilde{x}_3 = x_2 - a_3 = \frac{-5+\sqrt{33}}{8} \\
 & x_4 = 5 + \sqrt{33} \\
a_4 = 10 & \ldots
\end{array}$$

The b_i:

$$\begin{array}{ll}
b_0 = a_0 = 5 & b_0^2 \ \underline{\text{mod}}\ 33 = -8 \\
b_1 = a_0 a_1 + 1 = 6 & b_1^2 \ \underline{\text{mod}}\ 33 = 3 \\
b_2 = a_2 b_1 + b_0 = 17 & b_2^2 \ \underline{\text{mod}}\ 33 = -8.
\end{array}$$

Hence

$$x = b_0 b_2 = 5 \cdot 17 = 19 \bmod 33 \qquad \text{and} \qquad y = 8$$

so that

$$\gcd(x+y, n) = \gcd(27, 33) = 3 \qquad \text{and} \qquad \gcd(x-y, n) = \gcd(11, 33) = 11.$$

Example 2.2.19 Let us factorise $n = 197209$. Taking the b_i modulo n we get

$$\begin{array}{lll}
a_0 = 444 & b_0 = 444 & b_0^2 \ \underline{\bmod}\ n = -73 = -1 \cdot 73 \\
a_1 = 12 & b_1 = 5329 & b_1^2 \ \underline{\bmod}\ n = 145 = 5 \cdot 29 \\
a_2 = 6 & b_2 = 31428 & b_2^2 \ \underline{\bmod}\ n = -37 = -1 \cdot 37 \\
a_3 = 23 & b_3 = 159316 & b_3^2 \ \underline{\bmod}\ n = 720 = 2^4 \cdot 3^2 \cdot 5 \\
a_4 = 1 & b_4 = 191734 & b_4^2 \ \underline{\bmod}\ n = -143 = -11 \cdot 13 \\
a_5 = 5 & b_5 = 131941 & b_5^2 \ \underline{\bmod}\ n = 215 = 5 \cdot 43 \\
a_6 = 3 & b_6 = 193139 & b_6^2 \ \underline{\bmod}\ n = -656 = -2^4 \cdot 41 \\
a_7 = 1 & b_7 = 127871 & b_7^2 \ \underline{\bmod}\ n = 33 = 3 \cdot 11 \\
a_8 = 26 & b_8 = 165232 & b_8^2 \ \underline{\bmod}\ n = -136 = -2^3 \cdot 17 \\
a_9 = 6 & b_9 = 133218 & b_9^2 \ \underline{\bmod}\ n = 405 = 3^4 \cdot 5
\end{array}$$

Hence $x = b_3 \cdot b_9 = 126308 \bmod n$ and $y = 2^2 \cdot 3^3 \cdot 5 = 540$. Now $\gcd(x+y, n) = 991$ and $\gcd(x-y, n) = 199$.

Divertimento: solving the Pell equation

Let d be a positive square-free integer. The problem is to find integral solutions $x, y \in \mathbb{Z}$ to the equation

$$x^2 - dy^2 = 1$$

which is called the *Pell equation*. Here we show how this can be done using continued fractions.

First of all we consider the continued fraction of $\sqrt{d}$. We get real numbers $x_0, x_1, \ldots$ and integers $a_0, a_1, \ldots$ with

$$x_0 = \sqrt{d}, \quad a_i = \lfloor x_i \rfloor, \quad x_{i+1} = \frac{1}{x_i - a_i}.$$

Moreover we get the convergents $\frac{b_i}{c_i}$.

It is a fact, which we do not prove here, that the continued fraction of $\sqrt{d}$ is periodic. This means that there is an $s > 1$ such that $x_{s+1} = x_1$, which also implies $a_{s+1} = a_1$. We take $s > 1$ to be the smallest such number. This also means that $x_{ks+1} = x_1$ for $k = 1, 2, \ldots$. So since $x_i = a_1 + \frac{1}{x_{i+1}}$ we get

$$x_{ks} = a_{ks} + \frac{1}{x_1} = a_1 + \frac{1}{x_1}.$$

Setting $\tilde{x}_i = x_i - a_i$ we have, as seen in the proof of Theorem 2.2.11

$$\sqrt{d} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots \frac{1}{a_i + \tilde{x}_i}}}.$$

Now $a_{ks} + \tilde{x}_{ks} = a_{ks} + \frac{1}{x_1} = a_{ks} + x_0 - a_0 = a_{ks} - a_0 + \sqrt{d}$. Set $u = a_{ks} - a_0$, which lies in $\mathbb{Z}$. Then we see that we get $\sqrt{d}$ from the ks-th convergent $\frac{b_{ks}}{c_{ks}}$ by substituting $a_{ks} \mapsto u + \sqrt{d}$. For brevity we write $j_0 = ks$. Then $\frac{b_{ks}}{c_{ks}} = (a_{j_0} b_{j_0-1} + b_{j_0-2})/(a_{j_0} c_{j_0-1} + c_{j_0-2})$. Hence

$$\sqrt{d} = \frac{(u + \sqrt{d}) b_{j_0-1} + b_{j_0-2}}{(u + \sqrt{d}) c_{j_0-1} + c_{j_0-2}}.$$

Multiplying by the denominator on the right-hand side we get

$$(u c_{j_0-1} + c_{j_0-2})\sqrt{d} + d c_{j_0-1} = b_{j_0-1}\sqrt{d} + u b_{j_0-1} + b_{j_0-2},$$

yielding $u c_{j_0-1} + c_{j_0-2} = b_{j_0-1}$ and $u b_{j_0-1} + b_{j_0-2} = d c_{j_0-1}$. We multiply the first equation by b_{j_0-1}, the second by c_{j_0-1} and subtract. This gives

$$b_{j_0-1} c_{j_0-2} - b_{j_0-2} c_{j_0-1} = b_{j_0-1}^2 - d c_{j_0-1}^2.$$

By Theorem 2.2.9, the left-hand side of this equation is $(-1)^{j_0}$. We conclude:

- if s is even then the $(s-1)$-st convergent gives a solution to the Pell equation,
- if s is odd, then the $(2s-1)$-st convergent gives a solution to the Pell equation.

Example 2.2.20 Consider $d = 14$. Then for the continued fraction of $\sqrt{d}$ we get

i	a_i	x_{i+1}	b_i	c_i
0	3	$\frac{\sqrt{14}+3}{5}$	3	1
1	1	$\frac{\sqrt{14}+2}{2}$	4	1
2	2	$\frac{\sqrt{14}+2}{5}$	11	3
3	1	$\sqrt{14}+3$	15	4
4	6	$\frac{\sqrt{14}+3}{5}$	101	27

We see that $s = 4$, and therefore $x = 15$, $y = 4$ give a solution to the Pell equation. Indeed, $15^2 - 14 \cdot 4^2 = 1$.

The Pell equation has a long and interesting history. Apparently it was considered by Archimedes who posed the "cattle problem", whose solution goes via a Pell equation. It was also studied by the indian mathematician Brahmagupta (598 - 670), who was the first to give a method for solving it. Much later Fermat challenged other mathematicians to find a method to solve the equation. He accompanied his challenge with some particularly difficult cases; therefore it is very plausible that Fermat knew how to solve the equation. The challenge was mainly taken up by English mathematicians such as Brouncker and Wallis. Wallis published a book, also describing a method to solve the Pell equation, which was due to Brouncker. However, when Euler worked on the equation, he confused Brouncker with Pell, and called it the Pell equation. It is still referred to under his name, although Pell never considered it. Euler came up with a rudimentary form of the continued fraction method to solve the equation. Later Lagrange proved rigorously that this method is correct.

There are many interesting questions related to the Pell equation. One is how the solution (x, y) grows with d. For example, when $d = 139$ the smallest solution is $x = 77563250$, $y = 6578829$. For more information on this and other problems we refer to the book by Michael Jacobson and Hugh Williams, *Solving the Pell equation*, Springer Verlag 2009.

2.2.3 The elliptic curve method (ECM)

In 1999, Richard Brent (Factorization of the tenth Fermat number, *Math. Comp* 225), using ECM (Elliptic Curve Method) has found the factorisation of the Fermat number F_{10}

$$45592577 \cdot 6487031809 \cdot 4659775785220018543264560743076778192897 \cdot P_{252}$$

where P_{252} is a prime of 252 digits. (This factorisation now takes about 102 seconds on my laptop.)

In this section we describe this method for factoring integers that uses ellptic curves. It is due to H. W. Lenstra, Jr.

Elliptic curves

Here we let k be a field of characteristic not 2 or 3. Let $x^3 + ax + b \in k[x]$ be a polynomial with three distinct roots (possibly in some extension of k). Then

$$E(k) = \{(x, y) \in k^2 \mid y^2 = x^3 + ax + b\} \cup \{O\}$$

where O is a symbol, is called an *elliptic curve*.

Remark 2.2.21 Let $f(x) = x^3 + ax + b$. Then f has three distinct roots if and only if there does not exist $\alpha \in k$ (or in some extension of k) with $f(\alpha) = f'(\alpha) = 0$. Indeed, if f has three distinct roots $\alpha_1, \alpha_2, \alpha_3$ (lying in an extension of k) then $f' = (x - \alpha_1)(x - \alpha_2) + (x - \alpha_1)(x - \alpha_3) + (x - \alpha_2)(x - \alpha_3)$. Now $f(\alpha) = 0$ implies that α is one of the α_i, say $\alpha = \alpha_1$. But then $f'(\alpha) = (\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3) \neq 0$. Conversely, if f has a double root α then it is straightforward to see that $f(\alpha) = f'(\alpha) = 0$.

Let $P = (\alpha, \beta) \in E(k)$; we want to construct the line tangent to $E(k)$ in P. To compute the slope we consider the derivative of $y^2 = x^3 + ax + b$, that is,

$$2y\frac{dy}{dx} = 3x^2 + a = f'(x).$$

If f has three distinct roots then it is impossible that $\beta^2 = f(\alpha) = 0$ and $f'(\alpha) = 0$ at the same time. So the tangent line has a well defined slope, namely $\frac{3\alpha^2+a}{2\beta}$ ($\beta = 0$ means that the line is vertical).

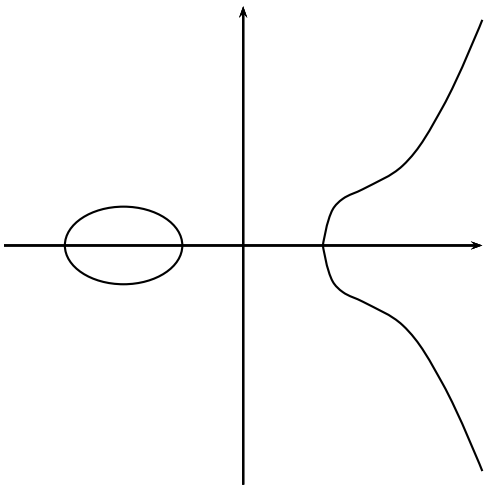
Write $x^3 + ax + b = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$ and define

$$\Delta = -\prod_{i<j}(\alpha_i - \alpha_j)^2$$

which is called the *discriminant* of the polynomial $x^3 + ax + b$. The latter has three distinct roots if and only if $\Delta \neq 0$. Furthermore, it is possible to show that $\Delta = 4a^3 + 27b^2$. (We will not go into that here.)

Conclusion: $y^2 = x^3 + ax + b$ defines an elliptic curve if and only if $\Delta = 4a^3 + 27b^2 \neq 0$.

If $k = \mathbb{R}$ we can make a graph of the curve. It will look like this:

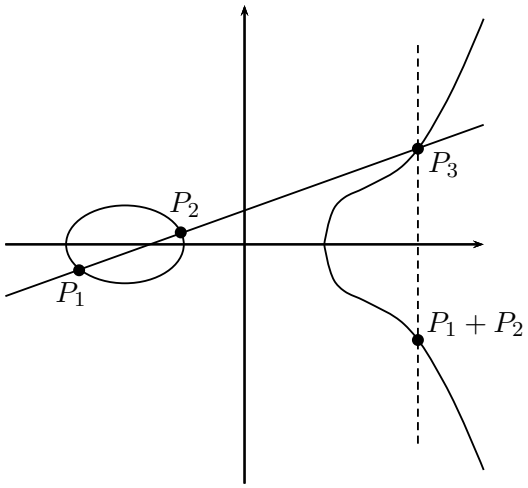


Note that the graph has to be symmetric with respect to the x-axis, as the equation is $y^2 = f(x)$. Hence $(x_0, y_0) \in E(k)$ implies $(x_0, -y_0) \in E(k)$.

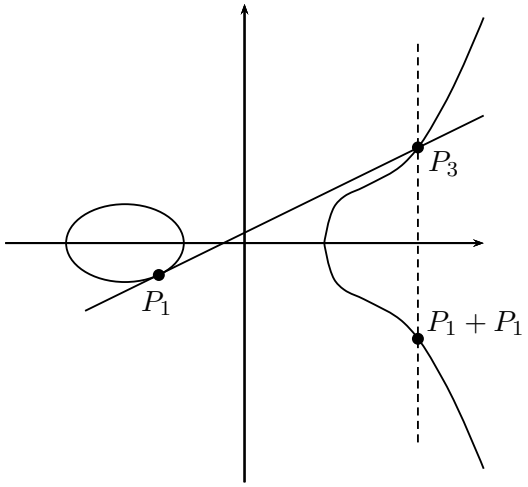
One of the main points of elliptic curves is that we can define an addition + on it that makes it into an abelian group. This is done as follows:

- The neutral element is O, i.e., $P + O = O + P = P$ for all $P \in E(k)$.

- If $P_1, P_2 \in E(k)$, $P_1 \neq P_2$ and $P_1, P_2 \neq O$ then we construct the line ℓ through P_1 and P_2. The line ℓ intersects $E(k)$ in a third point $P_3 = (x_3, y_3)$, and we set $P_1 + P_2 = (x_3, -y_3)$.



- If $P_1 = P_2 \in E(k)$, $P_1 \neq 0$, then ℓ will be the tangent line at $E(k)$ through P_1; after which we do the same thing.



- If the line ℓ is vertical then the sum will be O.

Theorem 2.2.22 *The set $E(k)$ with the operation $+$ is an abelian group.*

Here we do not prove this theorem; the difficult point is to show the associative law:

$$(P_1 + P_2) + P_3 = P_1 + (P_2 + P_3).$$

For a proof based on the Riemann-Roch theorem we refer to Joseph H. Silverman, *The Arithmetic of Elliptic Curves*, Springer Verlag, 1986. Below we will give explicit formulas for computing the sum of two points. It is also possible to use this formulas to prove that the addition is associative. However, this is rather cumbersome, as many special cases have to be considered, and requires the use of a computer. For details on that approach we refer to Stefan Friedl, *An elementary proof of the group law for elliptic curves*, Groups Complex. Cryptol. 9:117-123 (2017).

Addition formulas

Let $P_1, P_2 \in E(k)$, $P_1 \neq P_2$, $P_1, P_2 \neq O$ where $P_i = (x_i, y_i)$.

The line ℓ through P_1 and P_2 is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

Set $m = \frac{y_2 - y_1}{x_2 - x_1}$; we compute the points of intersection of ℓ and the curve $y^2 = x^3 + ax + b$. We get

$$(m(x - x_1) + y_1)^2 = x^3 + ax + b \quad \Longleftrightarrow \quad x^3 - m^2x^2 + ux + v = 0$$

for certain u and v.

Two solutions of this equation are x_1 and x_2 because P_1 and P_2 are on the curve and on the line. Hence

$$x^3 - m^2x^2 + ux + v = (x - x_1)(x - x_2)(x - x_3) = x^3 - (x_1 + x_2 + x_3)x^2 + \dots$$

so $x_1 + x_2 + x_3 = m^2$. Therefore

$$\begin{aligned} x_3 &= m^2 - x_1 - x_2 \\ y_3 &= m(x_3 - x_1) + y_1. \end{aligned}$$

It follows that $P_1 + P_2 = (m^2 - x_1 - x_2, -m(x_3 - x_1) - y_1)$.

We note that the sum does not depend on a and b. However, the points P_1 and P_2 of course do.

If $P_1 = P_2$ then we consider the tangent line ℓ. Take the derivative with respect to x of $y^2 = x^3 + ax + b$, i.e., $2y\frac{dy}{dx} = 3x^2 + a$. We get

- If $y_1 \neq 0$ then

$$m = \frac{3x_1^2 + a}{2y_1}$$

so the line ℓ has equation $y = m(x - x_1) + y_1$. It follows that $P_1 + P_1 = (x_3, -y_3)$ with

$$\begin{aligned} x_3 &= m^2 - 2x_1 \\ y_3 &= m(x_3 - x_1) + y_1. \end{aligned}$$

- If $y_1 = 0$ then $P_1 + P_1 = O$. (Note that it is impossible that $y_1 = 3x_1^2 + a = 0$.)

Example 2.2.23 Take $y^2 = x^3 + 17$ and $P_1 = (-1, 4)$. Then

$$\begin{aligned} m &= \frac{3(-1)^2 + 0}{8} = \frac{3}{8} \\ x_3 &= \frac{9}{64} + 2 = \frac{137}{64} \\ y_3 &= \frac{3}{8}\left(\frac{137}{64} + 1\right) + 4 = \frac{2651}{512}. \end{aligned}$$

So $P_1 + P_1 = \left(\frac{137}{64}, -\frac{2651}{512}\right)$.

Factorisation with elliptic curves

The main idea is to use the group law on an elliptic curve together with the reduction modulo p of its points.

Let p be a prime, and define

$$A_p = \left\{ \frac{\alpha}{\beta} \in \mathbb{Q} \mid p \text{ does not divide } \beta \right\}.$$

N.B. If p does not divide β then there exists $\gamma \in \mathbb{Z}$ with $\gamma\beta = 1 \bmod p$. Write $\gamma = \beta^{-1} \bmod p$.

Define $\psi_p \colon A_p \to \mathbb{F}_p$, with $\psi_p\left(\frac{\alpha}{\beta}\right) = \alpha\beta^{-1} \bmod p$. We have that A_p is a ring, and ψ_p is a ring homomorphism.

Let $a, b \in \mathbb{Z}$ such that $4a^3 + 27b^2 \neq 0 \bmod p$. Then

$$E(\mathbb{F}_p) = \{(x, y) \in \mathbb{F}_p^2 \mid y^2 = x^3 + ax + b\} \cup \{O\}$$

is an elliptic curve over $\mathbb{F}_p$.

Moreover,

$$E(\mathbb{Q}) = \{(x, y) \in \mathbb{Q} \mid y^2 = x^3 + ax + b\} \cup \{O\}$$

is an elliptic curve over $\mathbb{Q}$.

Define $\varphi_p\colon E(\mathbb{Q}) \to E(\mathbb{F}_p)$ in the following way. Let $P = \left(\frac{\alpha}{\beta}, \frac{\gamma}{\delta}\right) \in E(\mathbb{Q})$. If p does not divide β, δ then

$$\varphi_p(P) = \left(\psi_p\left(\frac{\alpha}{\beta}\right), \psi_p\left(\frac{\gamma}{\delta}\right)\right)$$

otherwise $\varphi_p(P) = O$.

Remark 2.2.24 Let $(\frac{u_1}{v_1}, \frac{u_2}{v_2}) \in E(\mathbb{Q})$, where $\gcd(u_1, v_1) = \gcd(u_2, v_2) = 1$ and $v_1, v_2 > 0$. Then

$$\frac{u_2^2}{v_2^2} = \frac{u_1^3 + au_1v_1^2 + bv_1^3}{v_1^3}.$$

If p is a prime dividing both v_1^3 and $u_1^3 + au_1v_1^2 + bv_1^3$ then p divides v_1 and hence also u_1. So we see that such primes do not exist, and hence $v_2^2 = v_1^3$. Writing $v_1 = p_1^{e_1} \cdots p_s^{e_s}$ (where the p_i are distinct primes) we see that all e_i are even and $v_2 = p_1^{\frac{3e_1}{2}} \cdots p_s^{\frac{3e_s}{2}}$. In other words, if we set $d = p_1^{\frac{e_1}{2}} \cdots p_s^{\frac{e_s}{2}}$ then we have $v_1 = d^2$ and $v_2 = d^3$. In particular we see that if a prime divides one denominator, it divides both.

Theorem 2.2.25 *The map φ_p is a group homomorphism. In other words, $\varphi_p(P_1 + P_2) = \varphi_p(P_1) + \varphi_p(P_2)$.*

For a complete proof of this theorem we refer to J. H. Silverman, J. Tate, *Rational Points on Elliptic Curves*, Springer Verlag, 2015, Section A.5. Here we only prove a special case.

Definition 2.2.26 *Let p be a prime and $P = \left(\frac{u_1}{v_1}, \frac{u_2}{v_2}\right) \in \mathbb{Q}^2$. Then we say that P is* good *for p if p does not divide v_1 or v_2.*

Lemma 2.2.27 *Let $E(k) = \{(x, y) \mid y^2 = x^3 + ax + b\} \cup \{O\}$ be an elliptic curve with $a, b \in \mathbb{Z}$ and $p > 2$ a prime not dividing $\Delta = 4a^3 + 27b^2$. Let $\varphi_p\colon E(\mathbb{Q}) \to E(\mathbb{F}_p)$ be the reduction modulo p. Let $P_1, P_2 \in E(\mathbb{Q})$ be two points that are good for p. Then $\varphi_p(P_1 + P_2) = \varphi_p(P_1) + \varphi_p(P_2)$.*

Proof. Let $P_i = (x_i, y_i)$. Here we also write $x \bmod p$ for $\psi_p(x)$.

- Suppose $P_1 \neq P_2$, $\varphi_p(P_1) \neq \varphi_p(P_2)$ and $x_1 \bmod p \neq x_2 \bmod p$. In this case one uses the same addition formulas over $\mathbb{Q}$ and over $\mathbb{F}_p$. Since mod p respects addition and multiplication, we get that $\varphi_p(P_1 + P_2) = \varphi_p(P_1) + \varphi_p(P_2)$. The same reasoning holds when $P_1 = P_2$. (Then of course $\varphi_p(P_1) = \varphi_p(P_2)$).

- Suppose that $P_1 \neq P_2$ and $x_1 \bmod p = x_2 \bmod p$. Write $x_2 = x_1 + p^r x$ with $x \in \mathbb{Q}$, $x \bmod p \neq 0$. Then $y_1^2 = y_2^2 \bmod p$ because $x_1^3 + ax_1 + b = x_2^3 + ax_2 + b \bmod p$ and hence $y_1 = \pm y_2 \bmod p$. We calculate

$$\begin{aligned} y_2^2 &= x_2^3 + ax_2 + b \\ &= (x_1 + p^r x)^3 + a(x_1 + p^r x) + b \\ &= x_1^3 + ax_1 + b + (3x_1^2 + a)p^r x + *p^{r+1} \\ &= y_1^2 + (3x_1^2 + a)p^r x + *p^{r+1} \end{aligned}$$

where $*$ is a rational number we don't want to write down.

Hence

$$(y_2 + y_1)(y_2 - y_1) = (3x_1^2 + a)p^r x + *p^{r+1}.$$

Now we distinguish three cases. In the first we have $y_1 = y_2 \bmod p$, and $y_2 + y_1 \neq 0 \bmod p$ In this case

$$y_2 - y_1 = (3x_1^2 + a)p^r x(y_2 + y_1)^{-1} + *p^{r+1}$$

for some $*$.

To compute $P_1 + P_2$ in $E(\mathbb{Q})$ we use

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3x_1^2 + a}{y_2 + y_1} + *p$$

as $x_2 - x_1 = p^r x$.

Hence $m \bmod p = \frac{3x_1^2+a}{2y_1} \bmod p$ which is exactly the m that is used for computing $\varphi_p(P_1)+\varphi_p(P_2)$. Hence

$$\varphi(P_1 + P_2) = \varphi_p(P_1) + \varphi_p(P_2).$$

In the second case we have $y_1 = -y_2 \bmod p$ and $y_2 - y_1 \neq 0 \bmod p$. Then

$$m = \frac{y_2 - y_1}{p^r x}.$$

Since $r > 0$ we get $\varphi_p(P_1 + P_2) = O$, which is equal to $\varphi_p(P_1) + \varphi_p(P_2)$.

In the third case we have $y_2 + y_1 = 0 \bmod p$ and $y_2 - y_1 = 0 \bmod p$. Then, since $p > 2$, we get $y_1 \bmod p = y_2 \bmod p = 0$. Hence $\varphi_p(P_1) = \varphi_p(P_2) = (\psi_p(x_1), 0)$. Hence $\varphi_p(P_1) + \varphi_p(P_2) = O$. Also $3x_1^2 + a \neq 0 \bmod p$, otherwise $p|\Delta$. So p^r is the highest power of p that divides $(y_2 + y_1)(y_2 - y_1)$. Since p divides both factors we get that $y_2 - y_1$ is only divisible by p^s for some $s < r$. Therefore, when computing the sum in $E(\mathbb{Q})$ we use an m of the form $\frac{a}{b}$ with $p|b$. It follows that $\varphi_p(P_1 + P_2) = O$. □

The idea lying behind the elliptic curve method for factorising integers is the following. The group $E(\mathbb{F}_p)$ is finite, so for $P \in E(\mathbb{F}_p)$ there exists k with

$$\underbrace{P + \ldots + P}_{k} = O.$$

Let $Q \in E(\mathbb{Q})$ with $\varphi_p(Q) = P$. Then by Theorem 2.2.25

$$\varphi_p(\underbrace{Q + \ldots + Q}_{k}) = \underbrace{P + \ldots + P}_{k} = O.$$

Hence $kQ = Q + \ldots + Q$ is equal to O or it has coordinates with a denominator divisible by p.

Algorithm 2.2.28 *Given: a composite number n.*
We compute: a factor of n

1. *We choose a curve $y^2 = x^3 + ax + b$, $a, b \in \mathbb{Z}$, with a point P on $E(\mathbb{Q})$.*
2. *Let $d = \gcd(n, 4a^3 + 27b^2)$. If $d = n$ we choose a different curve. If $d > 1$ and $d < n$ we have found a factor of n.*

3. Assume $d = 1$. Set $k = \mathrm{lcm}(2, 3, \ldots, K)$ for a certain K.

4. Compute $kP = \left(\frac{u_1}{v_1}, \frac{u_2}{v_2}\right)$. Let $d_i = \gcd(v_i, n)$. If d_i is bigger than 1, and smaller than n, then we have found a factor. Othewise we retry with a bigger K, or a different curve.

N.B. We know that n is not prime. The method works if

- (let p be a prime dividing n),

- if the order of $\varphi_p(Q)$ divides k then $\varphi_p(kQ) = k\varphi_p(Q) = O$; so if $kQ \neq O$ then p divides the denominator of u or of v.

Theorem 2.2.29 (Hasse) *We have*

$$p + 1 - 2\sqrt{p} \leq |E(\mathbb{F}_p)| \leq p + 1 + 2\sqrt{p}.$$

Remark 2.2.30 If $|E(\mathbb{F}_p)|$ divides k then we find p. For example when $K \geq p + 1 + 2\sqrt{p}$.

Example 2.2.31 Suppose that 61 divides n. Let E_a be the curve $y^2 = x^3 + ax + 3 - a$ and $P = (1, 2)$. We have

a	$\lvert E_a(\mathbb{F}_{61})\rvert$	order of (P mod 61)
-4	68	68
-3	54	9
-2	76	38
-1	69	69
0	48	12
1	68	4
2	66	33
3	72	6
4	67	67

Now $4P = O \in E_1(\mathbb{Q})$ so we cannot use E_1 with P. But $6P = \left(\frac{51825601}{16208676}, *\right) \in E_3(\mathbb{Q})$ and 61 divides 16208676.

Remark 2.2.32 It is enough to compute kP modulo n. Observe that if $rP = \left(\frac{u_1}{v_1}, \frac{u_2}{v_2}\right)$ and $\gcd(v_i, n) \neq 1$, then we have found a factor. Otherwise if $\gcd(v_i, n) = 1$ then there exist c_i, d_i with $c_i v_i + d_i n = 1$ and hence $\frac{u_i}{v_i} \bmod n = u_i c_i \bmod n$. This allows us to work with integer coordinates modulo n.

Example 2.2.33 Let's factorise $n = 39$. Let E be the curve given by $y^2 = x^3 + x - 1$ and $P = (1, 1)$ a point on it. We have $\Delta = 31$ and $\gcd(31, 39) = 1$. Take $k = 6$.

We have $2P = (2, -3)$. Let's compute $4P$:

$$m = \frac{3x_1^2 + a}{y_1} = \frac{12 + 1}{-6} = -\frac{13}{6}$$

and since $\gcd(6, 39) = 3$ we have found a factor.

Example 2.2.34 Now for something more difficult. Take $n = 185$ with the curve of Example 2.2.33 and $k = 8$. Again $\Delta = 31$ and $\gcd(31, 185) = 1$ so $y^2 = x^3 + x - 1$ defines an elliptic curve modulo every prime that divides n.

Also, $2P = (2, -3)$.

Now we compute $4P$. Again $m = -\frac{13}{6}$. Now

$$\begin{array}{rcrcr} 185 & = & 1 \cdot 185 & + & 0 \cdot 6 \\ 6 & = & 0 \cdot 185 & + & 1 \cdot 6 \\ \hline 5 & = & 1 \cdot 185 & - & 30 \cdot 6 \\ 1 & = & -1 \cdot 185 & + & 31 \cdot 6 \end{array}$$

hence $6^{-1} \bmod 185 = 31$. It follows that

$$m \bmod 185 = -13 \cdot 31 \bmod 185 = 152 \bmod 185.$$

So

$$\begin{aligned} x_3 &= m^2 - 2x_1 = 152^2 - 2 \cdot 2 \\ &= 160 \bmod 185 \\ y_3 &= m(x_3 - x_1) + y_1 = 152(160 - 2) - 3 \\ &= -37 \bmod 185. \end{aligned}$$

Hence $4P = (160, 37) \bmod 185$.

Now we compute $8P$. We have

$$m = \frac{3x_1^2 + a}{2y_1} = \frac{3 \cdot 160^2 + 1}{74}.$$

Furthermore,

$$\begin{array}{rcrcr} 185 & = & 1 \cdot 185 & + & 0 \cdot 74 \\ 74 & = & 0 \cdot 185 & + & 1 \cdot 74 \\ \hline 37 & = & 1 \cdot 185 & - & 2 \cdot 74 \\ 0 & = & -2 \cdot 185 & + & 5 \cdot 74 \end{array}$$

so $\gcd(185, 74) = 37$ and $185 = 5 \cdot 37$.

Here we note that the order of P in $E(\mathbb{Q})$ is ∞, while the order of P in $E(\mathbb{F}_{37})$ is 8.

Example 2.2.35 In his 1987 paper (Speeding the Pollard and Elliptic Curve Methods of Factorization, *Math. Comp.*), Peter Montgomery proposes to use elliptic curves $E_{\alpha,\beta}$ given by the equation

$$\beta y^2 = x^3 + \alpha x^2 + x.$$

The discriminant of the polynomial on the right is $\Delta = \alpha^2 - 4$. So we take $\alpha \neq \pm 2$, and of course $\beta \neq 0$.

First we derive the addition formulas on this curve. This goes in the same way as before. For $i = 1, 2$, let $P_i = (x_i, y_i)$ be points on $E_{\alpha,\beta}$. Suppose that $x_1 \neq x_2$ and set $m = \frac{y_2 - y_1}{x_2 - x_1}$, and let $y = mx + b$ be the equation of the line through P_1, P_2. Then intersecting this line with the curve leads to the equation $\beta(mx + b)^2 = x^3 + \alpha x^2 + x$, which is the same as $x^3 + (\alpha - \beta m^2)x^2 + *x + * = 0$. We know two solutions to this equation, namely x_1 and x_2. So

the polynomial on the left hand side is $(x-x_1)(x-x_2)(x-x_3) = x^3+(-x_1-x_2-x_3)x^2+\cdots$. Hence

$$x_3 = \beta m^2 - \alpha - x_1 - x_2$$

is the x-coordinate of the sum $P_3 := P_1 + P_2 = (x_3, y_3)$. Now we perform some manipulation, using $\beta y_i^2 = x_i^3 + \alpha x_i^2 + x_i$:

$$\begin{aligned} x_3(x_1-x_2)^2 &= \beta(y_1-y_2)^2 - (x_1-x_2)^2(\alpha+x_1+x_2) \\ &= -2\beta y_1 y_2 + x_1^2 x_2 + x_1 x_2^2 + 2\alpha x_1 x_2 + x_1 + x_2 \\ &= -2\beta y_1 y_2 + \frac{x_2}{x_1}(x_1^3 + \alpha x_1^2 + x_1) + \frac{x_1}{x_2}(x_2^3 + \alpha x_2^2 + x_2) \\ &= \beta \frac{(x_2 y_1 - x_1 y_2)^2}{x_1 x_2}. \end{aligned}$$

Set $P_4 = P_1 - P_2 = P_1 + (-P_2)$ and write $P_4 = (x_4, y_4)$. Since $-P_2 = (x_2, -y_2)$ we get $x_4(x_1-x_2)^2 = \beta(x_2 y_1 + x_1 y_2)^2/x_1 x_2$. We multiply the two formulas we obtained:

$$\begin{aligned} x_3 x_4 (x_1-x_2)^4 &= \beta^2 \frac{(x_2^2 y_1^2 - x_1^2 y_2^2)^2}{x_1^2 x_2^2} \\ &= \frac{(x_2^2(x_1^3 + \alpha x_1^2 + x_1) - x_1^2(x_2^3 + \alpha x_2^2 + x_2))^2}{x_1^2 x_2^2} \\ &= (x_1^2 x_2 + x_2 - x_1 x_2^2 - x_1)^2 \\ &= (x_1-x_2)^2(x_1 x_2 - 1)^2. \end{aligned}$$

The conclusion is

$$x_3 x_4 (x_1-x_2)^2 = (x_1 x_2 - 1)^2. \tag{2.2}$$

Next we consider the case where $P_1 = P_2$. In this case the x-coordinate of the sum $P_3 = P_1 + P_2 = 2P_1$ is

$$x_3 = \beta m^2 - \alpha - 2x_1 \text{ where } m = \frac{3x_1^2 + 2\alpha x_1 + 1}{2\beta y_1}.$$

Now on the one hand, $x_3 \cdot 4\beta y_1^2 = 4x_1 x_3(x_1^2 + \alpha x_1 + 1)$. On the other hand,

$$\begin{aligned} x_3 \cdot 4\beta y_1^2 &= (3x_1^2 + 2\alpha x_1 + 1)^2 - 8\beta x_1 y_1^2 - 4\alpha\beta y_1^2 \\ &= (3x_1^2 + 2\alpha x_1 + 1)^2 - (8x_1 - 4\alpha)(x_1^3 + \alpha x_1^2 + x_1) \\ &= (x_1^2 - 1)^2. \end{aligned}$$

So in this case we get

$$4x_1 x_3(x_1^2 + \alpha x_1 + 1) = (x_1^2 - 1)^2. \tag{2.3}$$

Let Q be a point of $E_{\alpha,\beta}(\mathbb{Q})$, and write the x-coordinate of kQ as $\frac{u_k}{d_k}$, where $u_k, d_k \in \mathbb{Z}$ (but not necessarily $\gcd(u_k, d_k) = 1$). Set $P_1 = Q_{k+1}$, $P_2 = Q_k$. Then $P_3 = P_1 + P_2 = Q_{2k+1}$ and $P_4 = P_1 - P_2 = P_1$. Then (2.2) implies that

$$\frac{u_{2k+1}}{d_{2k+1}} = \frac{d_1(u_{k+1}u_k - d_{k+1}d_k)^2}{u_1(u_{k+1}d_k - u_k d_{k+1})^2}$$

so we can set $u_{2k+1} = d_1(u_{k+1}u_k - d_{k+1}d_k)^2$ and $d_{2k+1} = u_1(u_{k+1}d_k - u_k d_{k+1})^2$. If we set $P_1 = kQ$, then from (2.3) we get

$$\frac{u_{2k}}{d_{2k}} = \frac{u_k^4 - 2u_k^2 d_k^2 + d_k^4}{4(u_k^3 d_k + \alpha u_k^2 d_k^2 + u_k d_k^3)},$$

so that $u_{2k} = (u_k^2 - d_k^2)^2$ and $d_{2k} = 4u_k d_k(u_k^2 + \alpha u_k d_k + d_k^2)$.

The point is that these formulas make it possible to compute $u_k \bmod n$, $d_k \bmod n$ rather quickly. Moreover, if the order of Q modulo p divides k, then p divides d_k and hence $\gcd(n, d_k)$ finds p. So we compute $d_k \bmod n$ for some k with many divisors, and compute $\gcd(n, d_k)$. If no divisor is found then we increase k, or try with another curve. This way the computation of various inverses modulo n is avoided.

For example, one can take $\alpha = a$ and $\beta = 4a + 10$, and $Q = (2, 1)$, for various a.

2.2.4 Complexity

Here we list some known facts about the complexity of the methods that we have seen (and some that we havan't seen), without bothering about the proofs.

- The algorithm trying division by $k = 2, 3, \ldots, \sqrt{n}$ has complexity $O(p)$ (order p), where p is the smallest prime factor of n. But, in the worst case, $O(p) = O(\sqrt{n})$ and $\sqrt{n} = n^{\frac{1}{2}} = \exp\left(\frac{1}{2}\log n\right)$ and $\log n \cong$ "number of digits of n" = size of n. So $\exp\left(\frac{1}{2}\log n\right)$ is exponential in the input size.

- The elliptic curve method has estimated complexity $O(\exp\sqrt{\log p \log(\log p)})$ where p is the smallest factor of n. This is called "subexponential".

- The continued fraction method has estimated complexity $O(\exp\sqrt{2\log p \log(\log p)})$.

- Two other famous methods are: the quadratic sieve with complexity $O(\exp\sqrt{\log n \log(\log n)})$, and the number field sieve, with complexity $O(\exp(\log p)^{1/3}(\log(\log p))^{2/3})$.

2.3 Primality proving with elliptic curves

In 1986 Goldwasser and Kilian published a method for proving that a given n, of which it has been established that it is prime with high probability, in fact is prime. It uses elliptic curves, and is based on the following theorem.

Theorem 2.3.1 *Let E be an elliptic curve with equation $y^2 = x^3 + ax + b$ with $a, b \in \mathbb{Z}$. Suppose that for $1 \le i \le r$ there are distinct prime numbers p_i and points $P_i \in E(\mathbb{Q})$, such that*

- *P_i is good for n, $1 \le i \le r$,*
- *$p_i P_i$ is not good for n, $1 \le i \le r$,*
- *$p_1 \cdots p_r > (n^{\frac{1}{4}} + 1)^2$.*

Then n is prime.

PROOF. Let p be a prime factor of n. Since the P_i are good for n, they are also good for p. So we get points $\varphi_p(P_i) \in E(\mathbb{F}_p)$. Moreover, $p_i\varphi_p(P_i) = O$ for all i. So $\varphi_p(P_i)$ has order p_i. Hence p_i divides $|E(\mathbb{F}_p)|$ and therefore so does $p_1 \cdots p_r$. So we get

$$(n^{\frac{1}{4}} + 1)^2 < p_1 \cdots p_r \leq |E(\mathbb{F}_p)| < p + 1 + 2\sqrt{p} = (\sqrt{p} + 1)^2$$

implying that $p > \sqrt{n}$. So the prime factors of n are greater that $\sqrt{n}$. We conclude that n is prime. □

Example 2.3.2 Let $n = 1873$ and let E be given by $y^2 = x^3 - 3x + 6$; and $P = (1, 2)$. Then $101P$ is not good for n. (Note that this can be established by doing some additions modulo n.) But $101 > (n^{\frac{1}{4}} + 1)^2 = 57.4\ldots$. It follows that n is prime.

The basic problem with this test is to find a suitable elliptic curve E and points P_i on it that are good for n, but for which there exist relatively small primes p_i such that p_iP_i are not good for n. (Note that the p_i need to be small with respect to n, otherwise the proof that the p_i are prime is as difficult as proving this for n.) In 1993 Atkin and Morain published a method for that, using the theory of complex multiplication. In practice this works surprisingly well, and huge numbers have been shown to be prime with this method.

2.4 Exercises

1. The smallest Carmichael number is $n = 561$. Show that it is not prime, using the Miller-Rabin test.

2. Factorise 203 and 899 with the method of Fermat.

3. In this exercise we find the factorisation of $n = 8616460799$ with Fermat's method. So we need to find $t > s$ such that $t^2 - s^2 = n$.

   (a) We have $n \bmod 3 = 2$. Show that $k^2 \bmod 3$ is 0 or 1 for $k \in \mathbb{Z}$. Prove that if $t^2 \bmod 3 = 1$, then $t^2 - n$ cannot be a square. Conclude that t has to be divisible by 3.

   (b) We have $n \bmod 8 = 7$. Show that $k^2 \bmod 8$ is 0,1,4. Prove that if $t^2 - n$ is a square, then $t^2 = 0 \bmod 8$. Conclude that t is divisible by 4.

   (c) Prove that t is divisible by 12.

   (d) Now try for t all multiples of 12, bigger than $\lceil\sqrt{n}\rceil$. Factorise n into a product of two smaller integers.

   (e) The English economist W. S. Jevons (1835-1882) wrote: "Given any two numbers, we may by a simple and infallible process obtain their product, but it is quite another matter when a large number is given, to determine its factors. Can the reader say what two numbers multiplied together produce the number 8616460799? I think it unlikely that anyone but myself will ever know."

4. Find the first 7 convergents of the continued fraction of $\sqrt{5}$.

5. Factorise 299, 341, 1537 and 1139 with the continued fraction method.

6. Let $a > 1$ be an integer and
$$\theta = \frac{a + \sqrt{a^2 + 4}}{2}.$$
Show that for the continued fraction of θ we have $a = a_0 = a_1 = a_2, \ldots$.

7. Let N be a positive integer that is not a square. For $i \geq 0$ let x_i and a_i be as in the algorithm for computing the continued fraction of $\sqrt{N}$. We consider the following statement:
$$x_i = \frac{u_i + \sqrt{N}}{v_i} \text{ where } u_i, v_i \in \mathbb{Z} \text{ satisfy } v_i \text{ divides } N - u_i^2.$$
   (a) Show that the statement is true for $i = 0$.
   (b) Suppose that the statement is true for some $i \geq 0$. Show that
$$x_{i+1} = \frac{v_i(a_i v_i - u_i + \sqrt{N})}{N - (a_i v_i - u_i)^2}.$$
   (c) Show that v_i divides $N - (a_i v_i - u_i)^2$.
   (d) Show that by setting $u_{i+1} = a_i v_i - u_i$, $v_{i+1} = \frac{N - u_{i+1}^2}{v_i}$ we obtain the statement for $i + 1$. (Remark: this gives a method for computing the u_i, v_i that is easy to implement on a computer.)
   (e) Show that $v_{i+2} = v_i - a_{i+1}^2 v_{i+1} + 2a_{i+1}u_{i+1}$ (so that we can avoid a division when computing v_{i+2}).

8. Let
$$x = \frac{1 + \sqrt{5}}{2}.$$
Let $a_0, a_1, \ldots$ be the integers corresponding to the continued fraction of x.
   (a) Prove that $a_i = 1$ for $i \geq 0$.
   (b) Let F_n be the n-th Fibonacci number (that is, $F_0 = 1$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$). Prove that the convergents of the continued fraction of x are $\frac{F_{n+1}}{F_n}$ for $n \geq 0$.

9. Consider the ellptic curve with equation $y^2 = x^3 + x - 1$. Let $P = (1, 1) \in E$. Compute $3P = P + P + P$.

10. In this exercise we factorise $n = 65 = 5 \cdot 13$ with the elliptic curve method. Let E be the elliptic curve with equation $y^2 = x^3 + 2x + 1$ and $P = (1, 2)$, a point.
   (a) Check that the equation defines an elliptic curve modulo every prime that divides n.
   (b) Compute all points of $E(\mathbb{F}_5)$ (here it can be handy to make a table of all squares modulo 5, and all $x^3 + 2x + 1$ for $x \in \mathbb{F}_5$). Show that $k\varphi_5(P) = O$ implies that k is divisible by 7. (Here $\varphi_5(P)$ is the reduction modulo 5 of P.)
   (c) Compute all points of $E(\mathbb{F}_{13})$. Prove that $4\varphi_{13}(P) = O$.
   (d) Compute $4P$ modulo n, and factorise n.

11. Again we consider $n = 65$, but now we take the elliptic curve E with equation $y^2 = x^3 + 3x + 2$ and point $P = (2, 4)$.

    (a) Check that the equation defines an elliptic curve modulo every prime that divides n.

    (b) Compute all points of $E(\mathbb{F}_5)$. Show that $k\varphi_5(P) = O$ if and only if k is divisible by 5.

    (c) Compute $5P$ modulo n and factorise n.

12. Factorise $n = 115$ with the elliptic curve method. (For example with the curve $y^2 = x^3 + 2x - 3$ and point $P = (2, 3)$.)

Chapter 3

Polynomial Factorisation

The problem considered in this chapter is to find the factorisation of a polynomial in $k[x]$, where k is a field.

The algorithms depend heavily on k. Here we treat algorithms for finite fields $k = \mathbb{F}_q$, with $q = p^n$, and $k = \mathbb{Q}$.

3.1 Some generalities on polynomials

Lemma 3.1.1 *For $f, g \in k[x]$ there exist unique $q, r \in k[x]$ with $\deg(r) < \deg(g)$ and $f = qg + r$ (division with remainder).*

Lemma 3.1.2 *Let $I \subset k[x]$ be a nonzero ideal. Then I is generated by a single element, that is, there is a $g \in k[x]$ with $I = \{fg \mid f \in k[x]\}$.*

Proof. Let $g \in I$ be a nonzero element of minimal degree. For $h \in I$ write $h = qg + r$ with $\deg(r) < \deg(g)$. This implies $r \in I$ whence $r = 0$ and $h = qg$. □

Lemma 3.1.3 *Let $f_1, f_2 \in k[x]$, not both equal to 0. Then there exists a unique monic $g \in k[x]$ with*

1) *g divides f_1, f_2;*

2) *if h divides f_1, f_2 then h divides g.*

Proof. Let $I \subset k[x]$ be the ideal generated by f_1, f_2, that is, $I = \{h_1 f_1 + h_2 f_2 \mid h_i \in k[x]\}$. Then by the previous lemma I is generated by a monic g (of minimal degree). Now $f_1, f_2 \in I$ hence g divides f_1 and f_2.

If h divides f_1, f_2 then $g = h_1 f_1 + h_2 f_2$ (since $g \in I = \langle f_1, f_2 \rangle$); we conclude that h divides g.

In order to show uniqueness, let $g' \in k[x]$ with the same properties as g. Then by 1) and 2), g' divides g. Analogously g divides g'. Since g and g' are monic we get $g = g'$. □

Definition 3.1.4 *The polynomial g from the previous lemma is called the* greatest common divisor *of f_1 and f_2.*

From the proof of Lemma 3.1.3 it follows that there exist h_1, h_2 such that $h_1 f_1 + h_2 f_2 = g$. In the same way as for the integers we have a euclidean algorithm for computing the greatest common divisor, and the h_1, h_2. It is based on the following lemma.

Lemma 3.1.5 *Write $f_1 = qf_2 + r$ with $\deg(r) < \deg(f_2)$. Then $\gcd(f_1, f_2) = \gcd(f_2, r)$.*

Proof. Set

$$D_1 = \{h \in k[x] \mid h \text{ divides } f_1, f_2\} \quad \text{and} \quad D_2 = \{h \in k[x] \mid h \text{ divides } f_2, r\}.$$

Then $D_1 = D_2$. So the elements of maximal degree of D_1 and D_2 coincide. □

To compute $\gcd(f_1, f_2)$ we replace (f_1, f_2) by (f_2, r) and so on. At a certain point we find the pair $(g, 0)$ and $\gcd(g, 0) = g$

Example 3.1.6 Let $f_1 = x^7 + 1$, $f_2 = x^4 + x^2 + x \in \mathbb{F}_2[x]$. Then

$$f_1 = (x^3 + x + 1)f_2 + x^3 + x + 1.$$

Set $f_3 = x^3 + x + 1$, then

$$f_2 = xf_3 + 0$$

hence $\gcd(f_1, f_2) = \gcd(f_2, f_3) = \gcd(f_3, 0) = f_3$.

N.B. A polynomial $h \in k[x]$ is called *irreducible* if $h \notin k$ and for every factorisation $h = ab$ with $a, b \in k[x]$ we have that a or b lies in k.

Lemma 3.1.7 *Let $a \in k[x]$ be irreducible and suppose that a divides bc where $b, c \in k[x]$. Then a divides b or a divides c.*

Proof. If a does not divide b then $\gcd(a, b) = 1$ so there exist h_1, h_2 with $h_1 a + h_2 b = 1$ whence $c = h_1 ac + h_2 bc$. Therefore a divides c. □

Theorem 3.1.8 *Let $f \in k[x]$, then there exist $c \in k$, irreducible monic and pairwise distinct $f_1, \ldots, f_r \in k[x]$ and $e_1, \ldots, e_r \in \mathbb{Z}_{>0}$ with $f = cf_1^{e_1} \cdots f_r^{e_r}$. Moreover, this factorisation is "essentially unique"; meaning that if we have another one $f = dg_1^{d_1} \cdots g_s^{d_s}$, then $c = d$, $r = s$, and after a permutation of the indices, $f_i = g_i$, $e_i = d_i$.*

Proof. Note that c is the coefficient of the highest degree monomial in f, hence it is uniquely determined. So without loss of generality we may assume that f is monic. If f is irreducible then there is nothing to prove. So suppose that $f = ab$ where $a, b \in k[x]$ are monic with $\deg(a), \deg(b) < \deg(f)$. By induction on the degree a and b are products of irreducible polynomials, hence so is f.

To show uniqueness suppose that $f = f_1^{e_1} \cdots f_r^{e_r} = g_1^{d_1} \cdots g_s^{d_s}$, where also the g_i are irreducible and monic.

By Lemma 3.1.7 f_1 divides a g_i. But g_i is irreducible and monic, hence $f_1 = g_i$. Cancelling both from the products, and by induction on the degree we conclude that the factorisation is esentially unique. □

Corollary 3.1.9 *Let $p, q, r \in k[x]$ with $\gcd(q, r) = 1$. Then $\gcd(p, qr) = \gcd(p, q)\gcd(p, r)$.*

PROOF. By unique factorization it follows that all divisors g of qr can be written as $g = h_1h_2$ where h_1 divides q and h_2 divides h_2. Let $d = \gcd(p, qr)$ and $d_1 = \gcd(p, q)$, $d_2 = \gcd(p, r)$. Then $d = g_1g_2$ where $g_1|q$, $g_2|r$. Since $g_1g_2|p$ we see that $g_1|p$, $g_2|p$. Hence $g_1|d_1$, $g_2|d_2$ so that $d|d_1d_2$. Conversely, it is obvious that d_1, d_2 divide d. As $\gcd(d_1, d_2) = 1$ this implies that $d_1d_2|d$. Since both d, d_1d_2 are monic it follows that they are equal. □

Problem. Given a monic $f \in k[x]$ find monic irreducible $f_1, \ldots, f_r$ such that $f = f_1^{e_1} \cdots f_r^{e_r}$.

Let $R_1, \ldots, R_s$ be rings. We recall the construction of the direct product: $R = R_1 \times \cdots \times R_s$ is the set consisting of $(r_1, \ldots, r_s)$ with $r_i \in R_i$; the ring operations are defined as follows

$$(r_1, \ldots, r_s) + (r'_1, \ldots, r'_s) = (r_1 + r'_1, \ldots, r_s + r'_s)$$

and

$$(r_1, \ldots, r_s)(r'_1, \ldots, r'_s) = (r_1r'_1, \ldots, r_sr'_s).$$

Then R is again a ring.

We denote the ideal generated by $f \in k[x]$ by $\langle f \rangle$, so $\langle f \rangle = \{gf \mid g \in k[x]\}$. Also, we write $[h]_f = h + \langle f \rangle$ for the coset of h modulo $\langle f \rangle$. If it is clear which f is meant, then we also write $[h]$. The polynomial h is called a *representative* of the coset $[h]_f$; note that a coset does not have a unique representative, indeed, all $h + gf$ are representatives of the same coset. If $f = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ then every coset has a unique representative of degree $< n$.

The quotient ring $k[x]/\langle f \rangle$ consists, by definition, of all cosets $[h]_f$ for $h \in k[x]$. They are added and multiplied by $[h_1] + [h_2] = [h_1 + h_2]$, $[h_1][h_2] = [h_1h_2]$. It is routine to show that these operations are well-defined, i.e., do not depend on the chosen representatives.

Theorem 3.1.10 (Chinese remainder theorem) *Let $f_1, f_2 \in k[x]$ be such that $\gcd(f_1, f_2) = 1$. Then there is an isomorphism*

$$\varphi : k[x]/\langle f_1f_2 \rangle \to k[x]/\langle f_1 \rangle \times k[x]/\langle f_2 \rangle,$$

defined by $\varphi([h]_{f_1f_2}) = ([h]_{f_1}, [h]_{f_2})$.

PROOF. First we show that φ is well-defined. Let $h' \in k[x]$ be such that $[h]_{f_1f_2} = [h']_{f_1f_2}$. Then $h' = h + gf_1f_2$, so $[h']_{f_i} = [h]_{f_i}$, $i = 1, 2$.

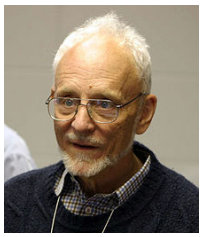
From the definitions of addition and multiplication in the various rings that appear, it is obvious that φ respects these operations.

We show that φ is injective. Suppose that $\varphi([h]_{f_1f_2}) = ([0]_{f_1}, [0]_{f_2})$. That means that h is divisible by both f_1 and f_2. Since $\gcd(f_1, f_2) = 1$ it follows that f_1f_2 divides h, so that $[h]_{f_1f_2} = [0]_{f_1f_2}$.

Finally we show that φ is surjective. There are polynomials $a, b \in k[x]$ with $af_1 + bf_2 = 1$. So $[af_1]_{f_2} = [1 - bf_2]_{f_2} = [1]_{f_2}$ and similarly $[bf_2]_{f_1} = [1]_{f_1}$. Let $h_1, h_2 \in k[x]$ and set $h = h_2af_1 + h_1bf_2$. Then $[h]_{f_1} = [h_1bf_2]_{f_1} = [h_1]_{f_1}[bf_2]_{f_1} = [h_1]_{f_1}$, and analogously, $[h]_{f_2} = [h_2]_{f_2}$. We conclude that $\varphi([h]_{f_1f_2}) = ([h_1]_{f_1}, [h_2]_{f_2})$. □

3.2 Berlekamp's algorithm

In this section we describe an algorithm due to Elwyn Berlekamp (1940-)



for factorising a polynomial over a finite field. Berlekamp is known, among other things, for his work in coding theory and game theory.

Definition 3.2.1 *Let $f \in k[x]$ be monic and $f = f_1^{e_1} \cdots f_r^{e_r}$ with f_i irreducible, monic and pairwise distinct. Then the $f_i^{e_i}$ are called the* primary factors *of f.*

In the following we write $h = g \bmod f$ to express that $h - g$ is divisible by f, or, in other words, $[h]_f = [g]_f$.

Lemma 3.2.2 *Let $f \in \mathbb{F}_q[x]$ be monic and $v \in \mathbb{F}_q[x]$ with $v^q = v \bmod f$. Then*

$$f = \prod_{a \in \mathbb{F}_q} \gcd(f, v - a).$$

Proof. In $\mathbb{F}_q[Y]$ we have

$$Y^q - Y = \prod_{a \in \mathbb{F}_q} (Y - a)$$

(because $a^q = a$ for all $a \in \mathbb{F}_q$). Substituting v for Y we obtain

$$v^q - v = \prod_{a \in \mathbb{F}_q} (v - a).$$

But $v^q - v$ is divisible by f, so $f = \gcd(f, v^q - v)$.

Observe that $\gcd(v - a, v - b) = 1$ if $a \neq b$, $a, b \in \mathbb{F}_q$. Indeed, if g divides $v - a$ and $v - b$ then g divides $b - a \neq 0$.

Using Corollary 3.1.9 we see that

$$\begin{aligned} f &= \gcd(f, v^q - v) \\ &= \gcd(f, \prod_{a \in \mathbb{F}_q} (v - a)) \\ &= \prod_{a \in \mathbb{F}_q} \gcd(f, v - a). \end{aligned}$$

□

Lemma 3.2.3 *Let $f \in \mathbb{F}_q[x]$ be monic and*

$$V = \{[h] \in \mathbb{F}_q[x]/\langle f\rangle \mid [h]^q = [h]\}.$$

Write $f = f_1^{e_1} \cdots f_r^{e_r}$ where the f_i are irreducible and distinct. Then V is a vector space over $\mathbb{F}_q$ of dimension r.

Note that $[h]^q = [h^q]$ so $[h]^q = [h]$ is the same as $h^q = h \bmod f$.

Proof. Let $v = [h], w = [g] \in V$, then $v + w = [h + g]$ and $(h+g)^q = h^q + g^q = h + g \bmod f$, so that $v + w \in V$. Furthermore, let $\alpha \in \mathbb{F}_q$, then $\alpha v = [\alpha h]$ and $(\alpha v)^q = \alpha^q v^q = \alpha v^q = \alpha v \bmod f$, whence $\alpha v \in V$. It follows that V is a vector space over $\mathbb{F}_q$.

By Theorem 3.1.10 there exists a ring isomorphism

$$\varphi\colon \mathbb{F}_q[x]/\langle f\rangle \to \mathbb{F}_q[x]/\langle f_1^{e_1}\rangle \times \cdots \times \mathbb{F}_q[x]/\langle f_r^{e_r}\rangle,$$

where $\varphi([h]_f) = ([h]_{f_1^{e_1}}, \ldots, [h]_{f_r^{e_r}})$.

Observe that by applying φ we get that $[h]_f^q = [h]_f$ is equivalent to $[h]_{f_i^{e_i}}^q = [h]_{f_i^{e_i}}$ for $1 \le i \le r$.

Let $[h]_f \in V$ then, as just seen, $h^q = h \bmod f_i^{e_i}$. By Lemma 3.2.2 it now follows

$$f_i^{e_i} = \prod_{a \in \mathbb{F}_q} \gcd(f_i^{e_i}, h - a). \tag{3.1}$$

But $h - a$ and $h - b$ are coprime when $a, b \in \mathbb{F}_q$ and $a \neq b$. Hence in (3.1) there is only one nontrivial factor $\gcd(f_i^{e_i}, h - a_i)$, for a unique $a_i \in \mathbb{F}_q$. Therefore $f_i^{e_i} = \gcd(f_i^{e_i}, h - a_i)$, implying that $f_i^{e_i}$ divides $h - a_i$. It follows that $h = a_i \bmod f_i^{e_i}$ and $\varphi([h]) = ([a_1]_{f_1^{e_1}}, \ldots, [a_r]_{f_r^{e_r}})$. So $\varphi(V) \subset \mathbb{F}_q \times \ldots \times \mathbb{F}_q$ where $\mathbb{F}_q \subset \mathbb{F}_q[x]/\langle f_i^{e_i}\rangle$ are the constant polynomials (more precisely: the cosets that have a representative in $\mathbb{F}_q$).

But $a^q = a$ for all $a \in \mathbb{F}_q$. So if $[h] \in \mathbb{F}_q[x]/\langle f\rangle$ is such that $\varphi([h]) = ([a_1]_{f_1^{e_1}}, \ldots, [a_r]_{f_r^{e_r}})$ where $a_i \in \mathbb{F}_q$, then $[h]^q = [h]$ and $[h] \in V$. Hence $\mathbb{F}_q \times \ldots \times \mathbb{F}_q \subset \varphi(V)$.

It follows that $\varphi(V) = \mathbb{F}_q \times \ldots \times \mathbb{F}_q$ (r factors), implying that V is a vector space of dimension r. □

Now we give Berlekamp's algorithm for finding the primary factors of a monic $f \in \mathbb{F}_q[x]$.

Algorithm 3.2.4 *Given: $f \in \mathbb{F}_q[x]$, monic.*
Compute: the primary factors of f.

1. *Let $\{v_1 = 1, v_2, \ldots, v_r\}$ be such that $\{[v_1], \ldots, [v_r]\}$ is a basis of $V = \{[h] \in \mathbb{F}_q[x]/\langle f\rangle \mid [h]^q = [h]\}$.*
2. *Set $P_1 = \{f\}$ and for $j = 2, \ldots, r$ let*

$$P_j = \bigcup_{h \in P_{j-1}} A_h, \text{ where } A_h = \{\gcd(h, v_j - a) \mid a \in \mathbb{F}_q\} \setminus \{1\}.$$

3. *Return P_r.*

Lemma 3.2.5 *The algorithm of Berlekamp finds the primary factors of f.*

PROOF. Write $P = P_r$. We have to show that

- f is the product of the elements in P;

- the elements of P are coprime;

- the elements of P are powers of irreducible polynomials.

First we claim that the product of the elements of P_j is f. This is certainly true for $j = 1$. Suppose the claim holds for P_{j-1} and write $P_{j-1} = \{h_1, \ldots, h_m\}$. Each h_i divides f, whence $v_j^q = v_j \bmod h_i$. So by Lemma 3.2.2 we see that h_i is the product of the elements in A_{h_i}. Hence by the induction hypothesis we get

$$f = h_1 \cdots h_m = \prod_{i=1}^{m} \prod_{g \in A_{h_i}} g = \prod_{g \in P_j} g.$$

The elements of P_j are pairwise coprime. We show this by induction on j. For $j = 1$ there is nothing to prove. Let $j > 1$ and write $P_{j-1} = \{h_1, \ldots, h_m\}$. Let $g_1, g_2 \in P_j$, $g_1 \neq g_2$, then $g_1 = \gcd(h_k, v_j - a)$, $g_2 = \gcd(h_l, v_j - b)$ for certain k, l and $a, b \in \mathbb{F}_q$. If $a \neq b$ then $v_j - a, v_j - b$ are coprime, and hence the same holds for g_1, g_2. If $a = b$ then $k \neq l$ and by induction h_k, h_l are coprime, and again the same follows for g_1, g_2.

Now let $h \in P$. Suppose that h is not a power of an irreducible. Then from what is said above it follows that h is divisible by $f_k^{e_k}$ and by $f_l^{e_l}$, $k \neq l$. To ease notation we assume that $k = 1$ and $l = 2$.

We fix a $j \geq 1$. Then for $j \leq l \leq r$ and $g \in P_l$ there exists $a_j \in \mathbb{F}_q$ (depending on g) with $v_j = a_j \bmod g$. Again we use induction to show this. Note that the elements of P_j are factors of $v_j - a$, for various $a \in \mathbb{F}_q$. Hence for $l = j$ the statement holds. Now let $g \in P_l$, $l > j$. Then g is a factor of a $\tilde{g} \in P_{l-1}$. By induction $v_j = a_j \bmod \tilde{g}$, hence also $v_j = a_j \bmod g$.

We apply this to $h \in P = P_r$ to conclude that there is $a_j \in \mathbb{F}_q$ with $v_j = a_j \bmod h$ for $1 \leq j \leq r$.

Now let $[v] \in V$ and write $[v] = \sum_{j=1}^{r} \beta_j [v_j]$, $\beta_j \in \mathbb{F}_q$. Since h divides f we get $v = \sum_j \beta_j v_j \bmod h$, and by the above

$$v = \sum_{j=1}^{r} \beta_j a_j \bmod h.$$

Set $a_v = \sum_{j=1}^{r} \beta_j a_j$; then $v = a_v \bmod h$, and $a_v \in \mathbb{F}_q$.

Since $f_1^{e_1}$, $f_2^{e_2}$ divide h we get $v = a_v \bmod f_1^{e_1}$ and $v = a_v \bmod f_2^{e_2}$. But as seen in the proof of Lemma 3.2.3, $\varphi\colon V \to \mathbb{F}_q \times \ldots \times \mathbb{F}_q$ is an isomorphism. But $\varphi(v) = ([a_v]_{f_1^{e_1}}, [a_v]_{f_2^{e_2}}, *, \ldots, *)$, in other words, the first two coordinates are always equal. This implies that φ is not surjective, and hence we have obtained a contradiction. □

Remaining problem. Given $f \in \mathbb{F}_q[x]$, $f = g^e$ with g irreducible, find g and e.

For this we compute $f' = \frac{d}{dx} f = e g' g^{e-1}$. There are two cases:

1) $f' = 0$. Then we claim that f is a polynomial in x^p where $q = p^m$, p a prime. Indeed, the exponents of x that occur in f are divisible by p. Hence $f = h(x^p) = h_1(x)^p$. We compute h_1, which again is a power of g, and continue with h_1.

2) $f' \neq 0$. Then $g = \frac{f}{\gcd(f, f')}$ since $\gcd(f, f') = g^{e-1}$.

Example 3.2.6 Let $f = x^4 + x^3 + x + 1 \in \mathbb{F}_2[x]$. Write $v \in \mathbb{F}_2[x]/\langle f\rangle$ as $v = a_0 + a_1x + a_2x^2 + a_3x^3$, $a_i \in \mathbb{F}_2$ (more precisely: the last expression is the unique representative of degree $< \deg(f)$ of a coset in $\mathbb{F}_2[x]/\langle f\rangle$).

Since $a_i^2 = a_i$ we get $v^2 = a_0 + a_1x^2 + a_2x^4 + a_3x^6$. Modulo f we have:

$$\begin{aligned} x^4 &= x^3 + x + 1 \\ x^5 &= x^4 + x^2 + x = x^3 + x + 1 + x^2 + x \\ &= x^3 + x^2 + 1 \\ x^6 &= x^4 + x^3 + x = x^3 + x + 1 + x^3 + x \\ &= 1 \end{aligned}$$

hence

$$\begin{aligned} v^2 &= a_0 + a_1x^2 + a_2(x^3 + x + 1) + a_3 \bmod f \\ &= a_0 + a_2 + a_3 + a_2x + a_1x^2 + a_2x^3 \bmod f \end{aligned}$$

and therefore

$$\begin{cases} a_0 + a_2 + a_3 = a_0 \\ a_2 = a_1 \\ a_1 = a_2 \\ a_2 = a_3 \end{cases} \iff a_1 = a_2 = a_3$$

so a basis of V is $\{v_1 = 1, x + x^2 + x^3\}$. So, in particular, $r = 2$.

In step $2a$) we replace f with

$$\begin{aligned} h_1 &= \gcd(f, x + x^2 + x^3) = x^2 + x + 1 \\ h_2 &= \gcd(f, 1 + x + x^2 + x^3) = x^2 + 1. \end{aligned}$$

So $P = \{h_1, h_2\}$. We have $h_1' = 1$ so that $\gcd(h_1, h_1') = 1$, thus h_1 is irreducible. Furthermore, $h_2' = 0$ implies $h_2 = (x + 1)^2$ and $(x + 1)' = 1$ so it is irreducible.

We conclude that $f = (x + 1)^2(x^2 + x + 1)$ is the factorisation of f.

3.3 The algorithm of Cantor-Zassenhaus

The algorithm of Berlekamp has the disadvantage to have to compute $\gcd(f, v - a)$ for $a \in \mathbb{F}_q$. If q gets large this is very laborious. Here we sketch a different aproach, due to David G. Cantor (1935-2012) and Hans Zassenhaus (1912-1991), based on the possibility of making random choices. For this we assume that q is odd.

Lemma 3.3.1 *Let $\gamma \in \mathbb{F}_q^*$, then $\gamma^{\frac{q-1}{2}} = \pm 1$. Moreover, $\gamma^{\frac{q-1}{2}} = 1$ if and only if γ is a square in $\mathbb{F}_q$ and $\gamma^{\frac{q-1}{2}} = -1$ if and only if γ is not a square.*

Proof. Let $\alpha \in \mathbb{F}_q^*$ be a primitive element: $\alpha^{q-1} = 1$, $\alpha^k \neq 1$ for $k < q - 1$. Then $\gamma = \alpha^i$, and

$$\gamma^{\frac{q-1}{2}} = \left(\alpha^{\frac{q-1}{2}}\right)^i.$$

We have that $\left(\alpha^{\frac{q-1}{2}}\right)^2 = 1$ implies $\alpha^{\frac{q-1}{2}} = \pm 1$ (there are only two roots of 1 in $\mathbb{F}_q$) but $\alpha^{\frac{q-1}{2}} \neq 1$ so $\alpha^{\frac{q-1}{2}} = -1$. Hence $\gamma^{\frac{q-1}{2}} = (-1)^i$.

Hence if γ is a square then $\gamma = \alpha^{2k}$, and hence $\gamma^{\frac{q-1}{2}} = (-1)^{2k} = 1$. If γ is not a square then $\gamma = \alpha^{2k+1}$ so that $\gamma^{\frac{q-1}{2}} = (-1)^{2k+1} = -1$. □

As before, we consider the space

$$V = \{[h] \in \mathbb{F}_q[x]/\langle f\rangle \mid [h]^q = [h]\}.$$

Write $r = \dim V$. If $r = 1$ then f is a power of an irreducible, and is therefore straightforward to factorise. So we suppose that $r \geq 2$.

Let $[h] \in V$, then

$$h^q - h = h\left(h^{\frac{q-1}{2}} - 1\right)\left(h^{\frac{q-1}{2}} + 1\right).$$

Note that these factors are coprime, so by Corollary 3.1.9,

$$f = \gcd(f, h^q - h) = \gcd(f, h)\gcd(f, h^{\frac{q-1}{2}} - 1)\gcd(f, h^{\frac{q-1}{2}} + 1).$$

This factorisation of f is trivial (that is, $f = f \cdot 1 \cdot 1$) if and only if f divides one of h, $h^{\frac{q-1}{2}} - 1$, $h^{\frac{q-1}{2}} + 1$.

We may assume that $\deg(h) < \deg(f)$ so f does not divide h.

Let $\varphi\colon \mathbb{F}_q[x]/\langle f\rangle \to \mathbb{F}_q[x]/\langle f_1^{e_1}\rangle \times \cdots \times \mathbb{F}_q[x]/\langle f_r^{e_r}\rangle$ be the isomorphism of Theorem 3.1.10, where the $f_i^{e_i}$ are the primary factors of f. Then as seen in the proof of Lemma 3.2.3, $\varphi(V) = \mathbb{F}_q \times \cdots \times \mathbb{F}_q$ (r factors); so $\varphi([h]) = ([a_1], \ldots, [a_r])$, $a_i \in \mathbb{F}_q$.

Now f divides $h^{\frac{q-1}{2}} - 1$ if and only if $h^{\frac{q-1}{2}} = 1 \bmod f$ and hence if and only if $\varphi([h])^{\frac{q-1}{2}} = ([1], \ldots, [1])$; or $a_i^{\frac{q-1}{2}} = 1$ for all i. By Lemma 3.3.1 we have that $a_i \in \mathbb{F}_q^*$ is a square for all i.

Analogously f divides $h^{\frac{q-1}{2}} + 1$ if and only if $a_i \in \mathbb{F}_q^*$ is not a square for all i.

Set

$$\begin{aligned} C &= \{(a_1, \ldots, a_r) \in \mathbb{F}_q^{*r} \mid a_i \text{ is a square } \forall i\} \\ &\cup \{(a_1, \ldots, a_r) \in \mathbb{F}_q^{*r} \mid a_i \text{ is not a square } \forall i\}. \end{aligned}$$

We have

$$|C| = \left(\frac{q-1}{2}\right)^r + \left(\frac{q-1}{2}\right)^r = 2\left(\frac{q-1}{2}\right)^r.$$

If we take $[h] \in V$ by random choice (where we use a uniform distribution) then the probability that we do not find a factorisation is

$$\mathcal{P}(\varphi([h]) \in C) = \frac{|C|}{|V|} = \frac{2\left(\frac{q-1}{2}\right)^r}{q^r} = 2\left(\frac{1 - \frac{1}{q}}{2}\right)^r < \frac{1}{2}.$$

So if k times we randomly choose a v the probability of not finding a factorisation is less than $\left(\frac{1}{2}\right)^k$.

Hans Zassenhaus (1912-1991)

made many contributions to abstract algebra. He was a pioneer in the area of computational algebra, using computers from the time when they were invented (i.e., the 50's). In computational algebra his main interest lay with computational number theory; he wrote a book about the subject together with Michael Pohst. But he also worked on algorithmic problems related to Lie algebras.

3.4 Factorisation of polynomials over $\mathbb{Q}$

First we show that the factorization of polynomials over $\mathbb{Q}$ is the same as the factorization of polynomials over $\mathbb{Z}$.

Definition 3.4.1 *Let $f \in \mathbb{Z}[x]$, $f = a_0 + a_1x + \ldots + a_nx^n$, $a_i \in \mathbb{Z}$. Then*

$$c(f) = \gcd(a_0, a_1, \ldots, a_n)$$

is called the content *of f.*

Lemma 3.4.2 *Let $f, g \in \mathbb{Z}[x]$ with $c(f) = c(g) = 1$, then $c(fg) = 1$.*

Proof. Suppose that $c(fg) > 1$. Then there exists a prime p dividing all coefficients of fg.

For $h \in \mathbb{Z}[x]$ we write $\bar{f}$ for the polynomial $h \bmod p$ which lies in $\mathbb{F}_p[x]$. Then $0 = \overline{fg} = \bar{f}\bar{g} \neq 0$ because $\bar{f} \neq 0 \neq \bar{g}$ as they have content 1. □

Lemma 3.4.3 (Gauss) *Let $f, g \in \mathbb{Z}[x]$. Then $c(fg) = c(f)c(g)$.*

Proof. Write $a = c(f)$, $b = c(g)$. Then $\frac{f}{a}, \frac{g}{b} \in \mathbb{Z}[x]$. But

$$fg = ab\frac{f}{a}\frac{g}{b} \quad \Rightarrow \quad c(fg) = c\left(ab\frac{f}{a}\frac{g}{b}\right) = ab \cdot c\left(\frac{f}{a}\frac{g}{b}\right).$$

Now $c\left(\frac{f}{a}\right) = c\left(\frac{g}{b}\right) = 1$ hence, by Lemma 3.4.2, $c\left(\frac{f}{a}\frac{g}{b}\right) = 1$ and therefore $c(fg) = ab$. □

Theorem 3.4.4 *Let $f \in \mathbb{Z}[x]$ and suppose that there exist $g, h \in \mathbb{Q}[x]$ with $f = gh$. Then there exist $a, b \in \mathbb{Q}$ such that $ag, bh \in \mathbb{Z}[x]$ and $f = (ag)(bh)$.*

In other words, every factorisation of f in $\mathbb{Q}[x]$ comes from a factorisation in $\mathbb{Z}[x]$.

Proof. First assume that $c(f) = 1$. Let $a, b \in \mathbb{Q}$ be such that $ag, bh \in \mathbb{Z}[x]$ and $c(ag) = c(bh) = 1$. We have

$$abf = (ag)(bh) \in \mathbb{Z}[x] \quad \text{and} \quad c(abf) = c(ag)c(bh) = 1.$$

Write $ab = \frac{s}{t}$ with $t \geq 1$; then all coefficients of f are divisible by t because $\frac{s}{t}f \in \mathbb{Z}[x]$. But $c(f) = 1$ so $t = 1$.

Now $c(abf) = c(sf) = 1$ hence $s = \pm 1$. Therefore $(ag)(bh) = \pm f$. If this is $-f$ then we replace a with $-a$ and obtain $f = (ag)(bh)$.

If $c(f) > 1$, then write $\gamma = c(f)$. We have that $\frac{1}{\gamma}f = (\frac{1}{\gamma}g)h$ and the content of $\frac{1}{\gamma}f$ is 1. So by the above, there are $a, b \in \mathbb{Q}$ such that $\frac{a}{\gamma}g$, bh lie in $\mathbb{Z}[x]$ and $\frac{1}{\gamma}f = \frac{a}{\gamma}g \cdot bh$. It follows that $f = ag \cdot bh$. □

So in order to factorise $f \in \mathbb{Q}[x]$ we can make some reductions

- assume $f \in \mathbb{Z}[x]$ (otherwise we multiply by an $s \in \mathbb{Z}$);

- the problem does not change if we look for factors in $\mathbb{Z}[x]$ (by Theorem 3.4.4);

- assume that f is square-free; otherwise we write $f = g^2 h$ and then $f' = 2gg'h + g^2 h'$ so g divides $\gcd(f, f')$; so we can factorise $\gcd(f, f')$ and $\frac{f}{\gcd(f,f')}$.

Idea. We factorise $f \bmod p$, p a prime, and we "lift" the factors to elements of $\mathbb{Z}[x]$. This process is called *Hensel lifting*.

3.4.1 Hensel lifting

Kurt Hensel (1861 - 1941)

invented p-adic numbers. The next theorem shows that a factorisation of a polynomial modulo p leads to a p-adic factorisation.

Theorem 3.4.5 (Hensel) *Let $f \in \mathbb{Z}[x]$ and p a prime, not dividing the leading coefficient of f. Let $g, h \in \mathbb{Z}[x]$ be such that*

- $\deg(g) + \deg(h) = \deg(f)$;

- $\gcd(g, h) = 1 \bmod p$;

- $f = gh \bmod p^k$ *for a certain integer* $k > 0$.

Then there exist $u, v \in \mathbb{Z}[x]$ with $\deg u < \deg g$, $\deg v \leq \deg h$; and after setting $\tilde{g} = g + p^k u$, $\tilde{h} = h + p^k v$, we have $\deg(\tilde{h}) = \deg h$, and

$$f = \tilde{g}\tilde{h} \bmod p^{k+1}.$$

Moreover, the u and v with these properties are uniquely determined modulo p.

Proof. Set $\tilde{g} = g + p^k u$, $\tilde{h} = h + p^k v$ for certain $u, v \in \mathbb{Z}[x]$. Then

$$f - \tilde{g}\tilde{h} = f - gh - p^k vg - p^k uh - p^{2k} uv$$

and this has to be 0 modulo p^{k+1}.

Observe that $f - gh$ is divisible by p^k.

So we want u, v such that

$$\frac{f - gh}{p^k} - vg - uh = 0 \bmod p.$$

Set $e = \frac{f-gh}{p^k}$.

We know that $\gcd(g,h) = 1 \bmod p$. So there exist $a, b \in \mathbb{Z}[x]$ with $ag + bh = 1 \bmod p$. Hence $eag + ebh = e \bmod p$.

So we could take $v = ea$ and $u = eb$ but this can increase the degree. The trick is to first adjust eb. Write $eb = qg + r$ with $\deg(r) < \deg(g)$. Then

$$eag + (qg + r)h = e \bmod p \quad \Leftrightarrow \quad (ea + qh)g + rh = e \bmod p.$$

Now we try $v = ea + qh \bmod p$ and $u = r \bmod p$. Of course, we are only interested in u and v modulo p.

With this choice we have $f = \tilde{g}\tilde{h} \bmod p^{k+1}$ and $\deg(u) < \deg(g)$. We need to check the degree of v.

Since $e = \frac{f-gh}{p^k}$ we get $\deg(e) \leq \deg(f)$. Moreover, $uh + vg = e \bmod p$ and $\deg(uh) < \deg(f)$ because $\deg(u) < \deg(g)$. Hence $\deg(vg \bmod p) \leq \deg(f)$, and therefore

$$\deg(v \bmod p) \leq \deg(h).$$

It also follows that $\deg(\tilde{h}) \leq \deg(h)$. However, if this inequality is strict then $f = \tilde{g}\tilde{h} \bmod p^{k+1}$ implies that p divides the leading coefficient of f, contrary to the assumption on p. Hence $\deg(\tilde{h}) = \deg(h)$.

In order to show uniqueness, let $u_0, v_0 \in \mathbb{Z}[x]$ have the same properties. Then $v_0 g + u_0 h = vg + uh \bmod p$, or

$$(u_0 - u)h = (v - v_0)g \bmod p.$$

Since g, h are coprime modulo p this implies that g divides $u_0 - u \bmod p$. But that is impossible because of the degrees; hence $u_0 - u = 0 \bmod p$. From that we also get $v - v_0 = 0 \bmod p$. □

Remark 3.4.6 Note that the $\tilde{g}$, $\tilde{h}$ in the conclusion of the preceding theorem satisfy the same properties as g and h, except that k has changed to $k + 1$. So we can continue and obtain a factorisation mod $p^{k+2}, p^{k+3}, \ldots$. In the limit this then yields a p-adic factorisation.

The proof of the theorem also gives a method for constructing $\tilde{g}, \tilde{h}$. One computes

- $e = \frac{f-gh}{p^k}$;
- a, b with $ag + bh = 1 \bmod p$ (extended euclidean algorithm);
- q, r with $eb = qg + r$ and $\deg(r) < \deg(g)$ (division with remainder);
- $u = r \bmod p$ and $v = ea + qh \bmod p$.

Then $\tilde{g} = g + p^k u$ e $\tilde{h} = h + p^k v$.

Example 3.4.7 Let $f = x^4 + 2x^3 - 3x^2 - 4x - 1$, $g = x^2 + 1$ and $h = x^2 + 2x + 2$. We have

$$gh = x^4 + 2x^3 + 3x^2 + 2x + 2 = f \bmod 3$$

hence

$$e = \frac{f - gh}{3} = -2x^2 - 2x - 1 = x^2 + x + 2 \bmod 3.$$

We do all computations modulo p because we are interested in u and v only modulo p. In particular e, a, b, q, v can be computed modulo p. We compute (everything modulo 3):

$$\begin{array}{rcrcrcl}
h & = & x^2+2x+2 & = & 1\cdot h & + & 0\cdot g \\
g & = & x^2+1 & = & 0\cdot h & + & 1\cdot g \\
\hline
 & & 2x+1 & = & h & - & g \\
 & & 2x^2+x & = & xh & - & xg \\
 & & x+1 & = & xh & + & (1-x)g \\
 & & 2 & = & (x+1)h & - & xg \\
 & & 1 & = & (2x+2)h & + & xg.
\end{array}$$

Hence $a = x$, $b = 2x + 2$ modulo 3. Therefore, modulo 3,

$$eb = (x^2 + x + 2)(2x + 2) = 2x^3 + x^2 + 1 = (2x + 1)g + x = qg + r.$$

So $u = r = x$ and

$$\begin{aligned} v &= ea + qh = (x^2 + x + 2)x + (2x + 1)(x^2 + 2x + 2) = 3x^3 + 6x^2 + 8x + 2 \\ &= 2x + 2 \bmod 3. \end{aligned}$$

We see that $\deg(v) < \deg(h) \bmod p$ but it is <u>not</u> guaranteed that v has degree $\leq \deg(h)$ without the modulo p operation.

Now

$$\begin{aligned} g_1 &= g + 3u = x^2 + 3x + 1 \\ h_1 &= h + 3(2x + 2) = x^2 + 8x + 8. \end{aligned}$$

Indeed we have

$$f - g_1h_1 = -9x^3 - 36x^2 - 36x - 9 = 0 \bmod 9.$$

For $k = 2$ we get

$$e = \frac{f - g_1h_1}{9} = -x^3 - 4x^2 - 4x - 1 = 2x^3 + 2x^2 + 2x + 2 \bmod 3.$$

Since $g_1 = g \bmod p$ and $h_1 = h \bmod p$ the same a and b of the previous step can also be used here. So

$$eb = x^4 + 2x^3 + 2x^2 + 2x + 1 = (x^2 + 2x + 1)g_1 + 0 = qg_1 + r.$$

We get $u = r = 0$ and

$$\begin{aligned} v &= ea + qh_1 = (2x^3 + 2x^2 + 2x + 2)x + (x^2 + 2x + 1)(x^2 + 8x + 8) \\ &= 2x + 2 \bmod 3. \end{aligned}$$

Now

$$\begin{aligned} g_2 &= g_1 + 9u = g_1 = x^2 + 3x + 1 \\ h_2 &= h_1 + 9(2x + 2) = x^2 + 26x + 26. \end{aligned}$$

And $f = g_2h_2 \bmod 27$. Continuing we find

$$\begin{aligned} g_3 &= x^2 + 3x + 1 \\ h_3 &= x^2 + 80x + 80. \end{aligned}$$

And $f = g_3h_3 \bmod 81$, and so on.

So, starting with a factorisation modulo p, we find factorisations modulo p^k, for any k we want. Now we describe how we can get the factorisation in $\mathbb{Z}[x]$ from the factorisation modulo p^k, when k is big enough.

Definition 3.4.8 *Let $f = \sum_{i=0}^{n} f_i x^i \in \mathbb{C}[x]$. The* norm *of f is*

$$||f|| = \sqrt{\sum_{i=0}^{n} |f_i|^2}.$$

Lemma 3.4.9 *Let $h \in \mathbb{C}[x]$, $a \in \mathbb{C}$. Then*

$$||(x-a)h|| = |a| \cdot ||(x - \bar{a}^{-1})h||.$$

Proof. Write $h = \sum_{i=0}^{m} h_i x^i$. We have

$$\begin{aligned}(x-a)h &= h_m x^{m+1} + (h_{m-1} - ah_m)x^m + \ldots + (h_0 - ah_1)x - ah_0 \\ &= \sum_{i=0}^{m+1} (h_{i-1} - ah_i)x^i\end{aligned}$$

where we set $h_{-1} = h_{m+1} = 0$.

In general we have

$$\begin{aligned}|u-v|^2 &= (u-v)(\bar{u}-\bar{v}) = u\bar{u} - u\bar{v} - \bar{u}v + v\bar{v} \\ &= |u|^2 - u\bar{v} - \bar{u}v + |v|^2.\end{aligned}$$

Hence

$$\begin{aligned}||(x-a)h||^2 &= \sum_{i=0}^{m+1} |h_{i-1} - ah_i|^2 \\ &= \sum (|h_{i-1}|^2 - \bar{a}h_{i-1}\bar{h}_i - a\bar{h}_{i-1}h_i + |ah_i|^2) \\ &= \sum (|h_i|^2 - \bar{a}h_{i-1}\bar{h}_i - a\bar{h}_{i-1}h_i + |\bar{a}h_{i-1}|^2) = \sum |\bar{a}h_{i-1} - h_i|^2.\end{aligned}$$

Because $\sum |h_{i-1}|^2 = \sum |h_i|^2$ and

$$|ah_i|^2 = |a|^2|h_i|^2 = |\bar{a}h_i|^2.$$

Therefore

$$\begin{aligned}||(x-a)h||^2 &= ||(\bar{a}x - 1)h||^2 \\ &= |\bar{a}|^2||(x - \bar{a}^{-1})h||^2 \\ &= |a|^2||(x - \bar{a}^{-1})h||^2.\end{aligned}$$

□

Theorem 3.4.10 (Landau-Mignotte) *Let $f \in \mathbb{Z}[x]$ and $g \in \mathbb{Z}[x]$ be a factor of f, $\deg(g) = m$. Write $g = \sum_{i=0}^{m} g_i x^i$. Then*

$$|g_i| \leq \binom{m}{i} ||f||.$$

PROOF. Let $b_1, \ldots, b_s$ and $a_1, \ldots, a_t$ in $\mathbb{C}$ be the zeros of f "with multiplicity" (so a zero of multiplicity k appears k times), where $|b_i| > 1$ and $|a_i| \leq 1$.

Write $f = \sum_{i=0}^{n} f_i x^i$. Then

$$f = f_n \prod_{i=1}^{s} (x - b_i) \prod_{j=1}^{t} (x - a_j)$$

so by Lemma 3.4.9

$$||f|| = ||f_n \prod_{i=1}^{s} (x - b_i) \prod_{j=1}^{t} (x - a_j)|| = |a_1 \cdots a_t| ||f_n \prod_{j=1}^{t} (x - \bar{a}_j^{-1}) \prod_{i=1}^{s} (x - b_i)||.$$

Write $h = \sum_i h_i x^i$, then $||h|| \geq |h_0|$ since $||h||^2 = \sum |h_i|^2$. Hence

$$||f|| \geq |a_1 \cdots a_t| |f_n| |\bar{a}_1^{-1} \cdots \bar{a}_t^{-1}| |b_1 \cdots b_s|.$$

For $z \in \mathbb{C}$, $|z\bar{z}^{-1}| = 1$, so we get

$$||f|| \geq |f_n b_1 \cdots b_s|.$$

Now let $\gamma_1, \ldots, \gamma_m$ be the zeros of g. Then $g = g_m(x - \gamma_1) \cdots (x - \gamma_m)$, whence

$$g_i = (-1)^{m-i} g_m \sigma_{m-i}(\gamma_1, \ldots, \gamma_m)$$

where $\sigma_i(x_1, \ldots, x_m)$ is the i-th elementary symmetric polynomial in $x_1, \ldots, x_m$; that is

$$\sigma_i(x_1, \ldots, x_m) = \sum_{1 \leq j_1 < j_2 < \ldots < j_i \leq m} x_{j_1} \cdots x_{j_i}.$$

We have that $\sigma_i(x_1, \ldots, x_m)$ is a sum of $\binom{m}{i}$ monomials of degree i.

We assume that the γ_i are ordered in such a way that $|\gamma_1| \geq |\gamma_2| \geq \ldots \geq |\gamma_m|$. Then

$$\begin{aligned} |\sigma_i(\gamma_1, \ldots, \gamma_m)| &\leq \binom{m}{i} |\gamma_1 \cdots \gamma_i| \\ &\leq \binom{m}{i} |b_1 \cdots b_s| \\ &\leq \binom{m}{i} \frac{||f||}{|f_n|} \end{aligned}$$

whence

$$|g_i| \leq |g_m| \binom{m}{m-i} \frac{||f||}{|f_n|} = |g_m| \binom{m}{i} \frac{||f||}{|f_n|}.$$

We have $|g_m| \leq |f_n|$ because g_m divides f_n. Therefore

$$|g_i| \leq \binom{m}{i} ||f||.$$

□

Lemma 3.4.11 *Let $a \in \mathbb{Z}$. Let $B \in \mathbb{Z}$, $B > 0$ be such that $|a| < B$. Let $M \in \mathbb{Z}$ be such that $M > 2B$ and let $a' \in (-\frac{M}{2}, \frac{M}{2}]$ be such that $a' \equiv a \bmod M$. Then $a' = a$.*

PROOF. We have $\frac{M}{2} > B$, hence $a \in (-\frac{M}{2}, \frac{M}{2}]$. But there is only one integer in that interval which is congruent to a modulo M. Hence $a = a'$. □

This immediately shows the following lemma.

Lemma 3.4.12 *Let $g \in \mathbb{Z}[x]$, $g = \sum_{i=0}^{m} g_i x^i$, and $B > 0$ with $|g_i| < B$. Let M be an integer with $M > 2B$ and write $g \bmod M = \sum_{i=0}^{m} \tilde{g}_i x^i$ where $\tilde{g}_i \in \left(-\frac{M}{2}, \frac{M}{2}\right]$. Then*

$$g = \sum_{i=0}^{m} \tilde{g}_i x^i$$

in other words: $g = g \bmod M$ and $g_i = \tilde{g}_i$.

Based on these results we have the following procedure for factorising a polynomial f in $\mathbb{Z}[x]$. We assume that f is square-free. We choose a prime p such that p does not divide the leading coefficient of f, and such that $f \bmod p$ is also square-free. For simplicity first we assume that $f \bmod p$ is the product of at most two irreducible factors.

- If $f \bmod p$ is irreducible then $f \in \mathbb{Z}[x]$ is irreducible, and we stop.
- Otherwise $f \bmod p = (g \bmod p)(h \bmod p)$, where $g \bmod p$ and $h \bmod p$ in $\mathbb{F}_p[x]$ are irreducible and distinct. (These factors can be found using Berlekamp's algorithm.)
- By Hensel lifting we find g_k and h_k such that $f = g_k h_k \bmod p^k$.
- Using the theorem of Landau-Mignotte we find a B so that the coefficients of a factor of f have absolute value $< B$.

Let $M = p^{k_0}$ with k_0 such that $M > 2B$. Then we find $\tilde{g}_i$, $\tilde{h}_i$ such that

$$\tilde{g} := g_{k_0} \bmod M = \sum \tilde{g}_i x^i, \qquad \tilde{g}_i \in \left(-\frac{M}{2}, \frac{M}{2}\right]$$
$$\tilde{h} := h_{k_0} \bmod M = \sum \tilde{h}_j x^j, \qquad \tilde{h}_i \in \left(-\frac{M}{2}, \frac{M}{2}\right].$$

Then $f = \tilde{g}\tilde{h}$ is the factorisation of f or f is irreducible. Indeed, if f is not irreducible then $f = \bar{g}\bar{h}$, for certain $\bar{g}, \bar{h} \in \mathbb{Z}[x]$. (We know that f has at most two factors over $\mathbb{Z}$, as this is the case modulo p.) Now $f \bmod p^{k_0} = g_{k_0} h_{k_0} \bmod p^{k_0} = (\bar{g} \bmod p^{k_0})(\bar{h} \bmod p^{k_0})$. However, by Theorem 3.4.5, the factorisation $f \bmod p^{k_0} = g_{k_0} h_{k_0} \bmod p^{k_0}$ is unique. Hence $\tilde{g} \bmod p^{k_0} = g_{k_0} \bmod p^{k_0} = \bar{g} \bmod p^{k_0}$ and $\tilde{h} \bmod p^{k_0} = h_{k_0} \bmod p^{k_0} = \bar{h} \bmod p^{k_0}$ (or the other way round, of course). So by Lemma 3.4.12, $\tilde{g} = \bar{g}$ and $\tilde{h} = \bar{h}$.

Example 3.4.13 As in Example 3.4.7, let $f = x^4 + 2x^3 - 3x^2 - 4x + 1$. Then $||f||^2 = 1 + 4 + 9 + 16 + 1 = 31$. Let $\alpha_2 x^2 + \alpha_1 x + \alpha_0$ be a factor of f. Then

$$|\alpha_2| \leq \binom{2}{2}\sqrt{31}, \quad |\alpha_1| \leq \binom{2}{1}\sqrt{31}, \quad |\alpha_0| \leq \binom{2}{0}\sqrt{31}$$

so $|\alpha_i| \leq 2\sqrt{31}$. Hence $|\alpha_i| \leq 11$, since $\alpha_i \in \mathbb{Z}$, so we can take $B = 11$.

We know that $f = g_3 h_3 \bmod 27$ where $g_3 = x^2 + 3x + 1$ and $h_3 = x^2 + 26x + 26$.

We choose $M = 27 > 2B = 22$. Now

$$\begin{aligned} g_3 \bmod M &= x^2 + 3x + 1 \\ h_3 \bmod M &= x^2 - x - 1 \end{aligned}$$

and $f = (x^2 + 3x + 1)(x^2 - x - 1)$; so we have factorised f.

3.4.2 Hensel lifting for more factors

Given $f \in \mathbb{Z}[x]$ and $f_1, \ldots, f_r \in \mathbb{Z}[x]$, and p a prime such that f_i, f_j are coprime modulo p. We assume that $\deg(f_1) + \ldots + \deg(f_r) = \deg(f)$, $f = f_1 \cdots f_r \bmod p$.

We compute $\tilde{f}_1, \ldots, \tilde{f}_r \in \mathbb{Z}[x]$ with $\tilde{f}_i = f_i \bmod p$ and $f = \tilde{f}_1 \cdots \tilde{f}_r \bmod p^k$ for a certain k.

1) Let $m = \lceil \frac{r}{2} \rceil$ and $g = f_1 \cdots f_m$, $h = f_{m+1} \cdots f_r$.

2) By Hensel lifting we compute $\tilde{g}, \tilde{h} \in \mathbb{Z}[x]$ with $\tilde{g} = g \bmod p$, $\tilde{h} = h \bmod p$ and $f = \tilde{g}\tilde{h} \bmod p^k$.

3) Recursively we compute $\tilde{f}_1, \ldots, \tilde{f}_r$ with $\tilde{f}_i = f_i \bmod p$, $\tilde{g} = \tilde{f}_1 \cdots \tilde{f}_m \bmod p^k$, $\tilde{h} = \tilde{f}_{m+1} \cdots \tilde{f}_r \bmod p^k$.

4) $f = \tilde{f}_1 \cdots \tilde{f}_r \bmod p^k$.

The algorithm

Now we describe a procedure for factorising a square-free $f \in \mathbb{Z}[x]$.

1) We choose a prime p such that $f \bmod p$ is square-free, and p does not divide the leading coefficient of f.

2) With Landau-Mignotte we compute a B such that

$$|\text{coefficient of a factor of } f| \leq B$$

and we choose k such that $M = p^k > 2B$.

3) Let $f_1, \ldots, f_r$ be the irreducible factors of $f \bmod p$ (obtained with Berlekamp's algorithm).

4) By Hensel lifting we get $\tilde{f}_1, \ldots, \tilde{f}_r \in \mathbb{Z}[x]$ such that $f = \tilde{f}_1 \cdots \tilde{f}_r \bmod p^k$ and $\tilde{f}_i = f_i \bmod p$.

5) For all $S \subset \{1, \ldots, r\}$ we compute

$$g_S = \prod_{i \in S} \tilde{f}_i \quad \text{and} \quad h_S = \prod_{i \notin S} \tilde{f}_i.$$

We write the coefficients of g_S and h_S modulo M in $\left(-\frac{M}{2}, \frac{M}{2}\right]$.

If $f = g_S h_S$ then we have found a factorisation. If for all S, $f_S \neq g_S h_S$ then f is irreducible.

Example 3.4.14 (Swinnerton-Dyer polynomials) Let $p_1 = 2, p_2 = 3, \ldots, p_n$ be the first n primes, and consider the field

$$F_n = \mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_n}).$$

Let f_n be the minimal polynomial of $\alpha_n = \sqrt{p_1} + \cdots + \sqrt{p_n}$. Using Galois theory one shows that f_n has degree 2^n and the roots of f_n are

$$\epsilon_1\sqrt{p_1} + \cdots + \epsilon_n\sqrt{p_n}$$

where $\epsilon_i \in \{-1, 1\}$. Using a bit of algebraic number theory one sees that $f_n \in \mathbb{Z}[x]$ (the $\sqrt{p_i}$ lie in the ring of integers of F_n, hence so does α_n; therefore $f_n \in \mathbb{Z}[x]$). Now let p be a prime, then all polynomials $x^2 - p_i$ have their roots in the quadratic extension $\mathbb{F}_{p^2}$ of $\mathbb{F}_p$. It follows that $f_n \bmod p$ factorises into linear factors over $\mathbb{F}_{p^2}$. Hence over $\mathbb{F}_p$, it factorises into linear or quadratic factors. So $f_n \bmod p$ has at least 2^{n-1} factors over $\mathbb{F}_p$. Now f_n, being the minimal polynomial of α_n, is irreducible in $\mathbb{Q}[x]$. But in order to show that using the algorithm outlined in this section one needs to inspect at least $2^{2^{n-1}}$ subsets S. So we see that at least for some polynomials this algorithm is not without its drawbacks.

3.5 Exercises

1. Prove that $x^4 + x + 1 \in \mathbb{F}_2[x]$ is irreducible.

2. Factorise $x^7 + 1 \in \mathbb{F}_2[x]$ and $x^4 + x + 1 \in \mathbb{F}_3[x]$. (In coding theory, the factorisation of $x^7 + 1 \in \mathbb{F}_2[x]$ implies that there are $8 = 2^3$ cyclic codes of length 7 over $\mathbb{F}_2$. The factorisation is used to find their generator polynomials; see J. H. van Lint, *Introduction to Coding Theory*, Chapter 6.)

3. Prove that $x^5 + x^3 + 1 \in \mathbb{Q}[x]$ is irreducible by factorising it modulo a suitably chosen prime.

4. Let $f = x^4 + 5x^3 + 6x^2 - x - 1$ and $g = x^2 + 1$, $h = x^2 + 2x + 2$. Then $g \bmod 3$ and $h \bmod 3$ are irreducible in $\mathbb{F}_3[x]$ and $f = gh \bmod 3$. Let g_k, h_k be the polynomials obtained from g, h by Hensel lifting. (So $g_1 = g$, $h_1 = h$, and $f = g_k h_k \bmod 3^k$.)

   (a) Compute g_k, h_k for $k = 2, 3$.

   (b) Prove that $g_k = x^2 + 3x + 1$ and $h_k = x^2 + 2x + 3^k - 1$ for $k \geq 2$.

   (c) Show that the coefficients of a factor of f are ≤ 24 in absolute value. Factorise f.

5. Let $f = x^5 + 2x^3 + 2x^2 + x + 1$. Factorise $f \bmod 3$, and prove that $f \in \mathbb{Q}[x]$ is irreducible.

6. In this exercise we show that $f = x^4 + 1 \in \mathbb{Q}[x]$ is irreducible, but $f \bmod p$ factorises modulo every prime p.

   (a) Factorise $f \bmod 2$ (hint: it is easy to find a root of f).

   (b) Let p be an odd prime. Let V be the space of $v \in \mathbb{F}_p[x]/\langle f \rangle$ such that $v^p = v \bmod f$. Compute a basis of V, treating the cases $p = 1+4k$, k even and odd, and $p = 3+4k$, k even and odd, separately. Conclude that $f \bmod p$ factorises for all primes p.

   (c) Factorise $f \bmod 5$.

(d) By Hensel lifting find $g, h \in \mathbb{Z}[x]$ with $f = gh \bmod 125$.

(e) Find a bound on the coefficients of a possible factor of f in $\mathbb{Z}[x]$ (Landau-Mignotte). Prove that f is irreducible.

Chapter 4

Lattice Basis Reduction

Here we describe the LLL algorithm, named after its inventors, A.K. Lenstra, H. W. Lenstra Jr. and L. Lovász (Factoring polynomials with rational coefficients, *Mathematische Annalen*, 261, 515-534 (1982)).

The original application of these authors was to polynomial factorisation. Nowadays lattice basis reduction finds many applications in computational problems of pure mathematics (computational number theory), and it is also used in real world applications such as the Global Positioning System GPS (for an overview we refer to the article Lattice Reduction by D. Wübben, D. Seethaler, J. Jaldén and G. Matz, *IEEE Signal Processing Magazine*, **70**, 2011). In the last section of this chapter we discuss an application in cryptography.

Notation: we work in the real vector space $\mathbb{R}^n$. For an element $x \in \mathbb{R}^n$, its i-th coordinate is written $x(i)$, so $x = (x(1), \ldots, x(n))$. The standard inner product, for $x, y \in \mathbb{R}^n$, is $(x, y) = \sum_{i=1}^{n} x(i)y(i)$. The vectors $x, y \in \mathbb{R}^n$ are said to be *orthogonal* if $(x, y) = 0$. The *norm* of an $x \in \mathbb{R}^n$ is $\|x\| = \sqrt{(x, x)}$. For $x_1, \ldots, x_k \in \mathbb{R}^n$ we denote their linear span by $\langle x_1, \ldots, x_k \rangle_L$.

4.1 Lattices

A *lattice* in $\mathbb{R}^n$ is the $\mathbb{Z}$-span of a basis of it. More in detail: a lattice in $\mathbb{R}^n$ is a set $L = \{m_1x_1 + \cdots + m_nx_n \mid m_i \in \mathbb{Z}\}$, where $x_1, \ldots, x_n$ is a basis of $\mathbb{R}^n$. One of the main problems concerning lattices is that they can contain rather "short" vectors, although a basis may be formed by rather "long" vectors. The ability to find short vectors in a lattice has many applications.

Here we write elements of $\mathbb{R}^n$ as row vectors. A basis $x_1, \ldots, x_n$ of $\mathbb{R}^n$ corresponds to the matrix X whose rows are the x_i. Now let L be as above, and let Y be a second basis of L. Then there is an $n \times n$-integral matrix C with $\det(C) = \pm 1$, such that $Y = CX$ (the proof of this is left as an exercise). Conversely, if C is an $n \times n$-matrix with integer entries and determinant ± 1 and $Y = CX$, then the rows of Y, $y_1, \ldots, y_n$ also form a basis of L.

Example 4.1.1 Let $L \subset \mathbb{R}^3$ be the lattice spanned by

$$x_1 = (4629703, 1594165, 2781628),\ x_2 = (-1641, -565, -986),\ x_3 = (37652, 12964, 22623).$$

Let X be the matrix with rows x_1, x_2, x_3, and

$$A = \begin{pmatrix} 1579935 & -2846 & -53628 \\ -560 & 1 & 19 \\ 12849 & -23 & -436 \end{pmatrix},\quad Y = \begin{pmatrix} 3 & 1 & 2 \\ 1 & -5 & 1 \\ 2 & 0 & 7 \end{pmatrix}.$$

Then $\det(A) = 1$ and $X = AY$. So the rows of Y also form a basis of L. The basis X is "bad" since it consists of vectors of large length. With respect to X, the basis Y is very "good". The question is: given a basis like X, how do we find a "good" basis? Here it is even not immediately clear how we can decide whether a given other basis is "good" (maybe it is just "better", but not yet "good"). The LLL algorithm succeeds in finding a "reasonably good" basis, and it also quantifies what is meant by "reasonably good". For example, using the LLL algorithm on the basis X above, yields the basis

$$\begin{pmatrix} 3 & 1 & 2 \\ -1 & -1 & 5 \\ 1 & -5 & 1 \end{pmatrix}.$$

4.2 Properties of Gram-Schmidt orthogonalisation

Let $U \subset \mathbb{R}^n$ be a subspace. Then $U^\perp = \{x \in \mathbb{R}^n \mid (x, u) = 0 \text{ for all } u \in U\}$ is called the *orthogonal complement* of U. Every $x \in \mathbb{R}^n$ can uniquely be written as $x = u + u^\perp$, where $u \in U$, $u^\perp \in U^\perp$. Indeed, $U \cap U^\perp = 0$, and $\dim U + \dim U^\perp = n$, so a basis of $\mathbb{R}^n$ can be formed by concatenating a basis of U and a basis of $U^\perp$. The vectors u and $u^\perp$ are called the projection of x on U and on $U^\perp$ respectively.

Proposition 4.2.1 *Let $x_1, \ldots, x_n$ be a basis of $\mathbb{R}^n$. For $1 \le j \le n$ let x_j^* be the projection of x_j on $\langle x_1, \ldots, x_{j-1}\rangle_L^\perp$ (so $x_1^* = x_1$). Then*

1. *$\langle x_1, \ldots, x_j\rangle_L = \langle x_1^*, \ldots, x_j^*\rangle_L$ for $1 \le j \le n$.*
2. *$(x_i^*, x_j^*) = 0$ for $1 \le i < j \le n$.*
3. $$x_j^* = x_j - \sum_{i=1}^{j-1} \mu_{ji} x_i^*, \text{ where } \mu_{ji} = \tfrac{(x_j, x_i^*)}{(x_i^*, x_i^*)}. \tag{4.1}$$

Proof.

1. Here we use induction on j, the statement for $j = 1$ being trivial. Let $U = \langle x_1, \ldots, x_j\rangle_L$. Then $x_{j+1} = u + x_{j+1}^*$ for a $u \in U$. So $\langle x_1, \ldots, x_j, x_{j+1}\rangle_L = \langle x_1, \ldots, x_j, x_{j+1}^*\rangle_L = \langle x_1^*, \ldots, x_j^*, x_{j+1}^*\rangle_L$ (where the last equality follows from the induction hypothesis).
2. This follows immediately from 1. along with the definition of x_j^*.

3. From 1. and the definition of x_j^* it follows that there are μ_{ji} with $x_j^* = x_j - \sum_{i=1}^{j-1} \mu_{ji} x_i^*$. Let $1 \leq i_0 \leq j-1$, then from 2. we have $0 = (x_{i_0}^*, x_j^*) = (x_{i_0}^*, x_j) - \mu_{j,i_0}(x_{i_0}^*, x_{i_0}^*)$. So we have 3.

□

Definition 4.2.2 *The basis $x_1^*, \ldots, x_n^*$ defined in Proposition 4.2.1 is called the* Gram-Schmidt orthogonalisation *of the basis $x_1, \ldots, x_n$. The μ_{ji} are called the* Gram-Schmidt coefficients *(GS-coefficients, for short).*

For the rest of this section we let $x_1, \ldots, x_n$ be a basis of $\mathbb{R}^n$, with Gram-Schmidt orthogonalisation $x_1^*, \ldots, x_n^*$. By μ_{ji} we denote the GS-coefficients, for $1 \leq i < j \leq n$. For convenience we set $\mu_{jj} = 1$ for all j and $\mu_{ji} = 0$ if $j < i$. Also, for $1 \leq k \leq n$ we let X_k be the $k \times n$-matrix with rows $x_1, \ldots, x_k$. Similarly, we let X_k^* be the matrix with rows $x_1^*, \ldots, x_k^*$. By M_k we denote the $k \times k$-matrix whose (j,i)-th entry is μ_{ji}. Then (4.1) translates to

$$X_k = M_k X_k^*. \tag{4.2}$$

Also we consider the $k \times k$-matrix G_k whose (i,j)-entry is (x_i, x_j). It is called the *Gram matrix* of $x_1, \ldots, x_k$. We have that G_k is symmetric, and $G_k = X_k X_k^T$. The k-th Gram determinant is $d_k = \det(G_k)$.

Lemma 4.2.3 $d_k = \|x_1^*\|^2 \cdots \|x_k^*\|^2$.

Proof. Using (4.2) and $\det(M_k) = 1$, we compute: $d_k = \det(G_k) = \det(X_k X_k^T) = \det((M_k X_k^*)(M_k X_k^*)^T) = \det(M_k (X_k^* (X_k^*)^T) M_k^T) = \det(X_k^* (X_k^*)^T) = \|x_1^*\|^2 \cdots \|x_k^*\|^2$ (since the x_i^* are orthogonal). □

The last statement of the next lemma is not used in the sequel, so can be skipped. (However, it is relevant to a certain exercise.)

Lemma 4.2.4 *Let j be an integer with $1 \leq j < n$. Define $\hat{x}_i = x_i$ if $i \neq j, j+1$, and $\hat{x}_j = x_{j+1}$, $\hat{x}_{j+1} = x_j$. Let $\hat{x}_1^*, \ldots, \hat{x}_n^*$ denote the Gram-Schmidt orthogonalisation of $\hat{x}_1, \ldots, \hat{x}_n$. Then $\hat{x}_i^* = x_i^*$ if $i \neq j, j+1$, $\hat{x}_j^* = x_{j+1}^* + \mu_{j+1,j} x_j^*$, and*

$$\hat{x}_{j+1}^* = \frac{\|x_{j+1}^*\|^2}{\|\hat{x}_j^*\|^2} x_j^* - \mu_{j+1,j} \frac{\|x_j^*\|^2}{\|\hat{x}_j^*\|^2} x_{j+1}^*.$$

Proof. If $i \neq j, j+1$ then $\langle \hat{x}_1, \ldots, \hat{x}_{i-1} \rangle_L = \langle x_1, \ldots, x_{i-1} \rangle_L$, and $\hat{x}_i = x_i$, so then the result

follows from the definition of x_i^* and $\hat{x}_i^*$. Now we compute

$$\begin{aligned}
\hat{x}_j^* &= \hat{x}_j - \sum_{i=1}^{j-1} \frac{(\hat{x}_j, \hat{x}_i^*)}{(\hat{x}_i^*, \hat{x}_i^*)} \hat{x}_i^* \\
&= x_{j+1} - \sum_{i=1}^{j-1} \frac{(x_{j+1}, x_i^*)}{(x_i^*, x_i^*)} x_i^* \\
&= x_{j+1} - \sum_{i=1}^{j-1} \mu_{j+1,i} x_i^* \\
&= x_{j+1} - \sum_{i=1}^{j} \mu_{j+1,i} x_i^* + \mu_{j+1,j} x_j^* \\
&= x_{j+1}^* + \mu_{j+1,j} x_j^*.
\end{aligned}$$

Now we derive some consequences of this. First of all, since the x_i^* are orthogonal,

$$\|\hat{x}_j^*\|^2 = \|x_{j+1}^*\|^2 + \mu_{j+1,j}^2 \|x_j^*\|^2. \tag{4.3}$$

Secondly $(x_j, \hat{x}_j^*) = (x_j, x_{j+1}^*) + \mu_{j+1,j}(x_j, x_j^*) = \mu_{j+1,j}(x_j, x_j^*) = \mu_{j+1,j} \sum_{i=1}^{j} \mu_{j,i}(x_i^*, x_j^*) = \mu_{j+1,j}(x_j^*, x_j^*)$, so

$$(x_j, \hat{x}_j^*) = \mu_{j+1,j} \|x_j^*\|^2. \tag{4.4}$$

And now:

$$\begin{aligned}
\hat{x}_{j+1}^* &= \hat{x}_{j+1} - \sum_{i=1}^{j} \frac{(\hat{x}_{j+1}, \hat{x}_i^*)}{(\hat{x}_i^*, \hat{x}_i^*)} \hat{x}_i^* \\
&= x_j - \sum_{i=1}^{j} \frac{(x_j, \hat{x}_i^*)}{(\hat{x}_i^*, \hat{x}_i^*)} \hat{x}_i^* \quad (\text{as } \hat{x}_{j+1} = x_j) \\
&= x_j^* - \frac{(x_j, \hat{x}_j^*)}{(\hat{x}_j^*, \hat{x}_j^*)} \hat{x}_j^* \quad (\text{as, for } i < j,\ \hat{x}_i^* = x_i^*) \\
&= x_j^* - \frac{\mu_{j+1,j} \|x_j^*\|^2}{\|\hat{x}_j^*\|^2} (x_{j+1}^* + \mu_{j+1,j} x_j^*) \quad (\text{by (4.4), and the result for } \hat{x}_j^*) \\
&= \frac{\|x_{j+1}^*\|^2}{\|\hat{x}_j^*\|^2} x_j^* - \mu_{j+1,j} \frac{\|x_j^*\|^2}{\|\hat{x}_j^*\|^2} x_{j+1}^* \quad (\text{by (4.3)}).
\end{aligned}$$

□

Lemma 4.2.5 *Let $y = m_1 x_1 + \cdots + m_n x_n$, where $m_i \in \mathbb{Z}$. Then $\|y\| \geq \min(\|x_1^*\|, \ldots, \|x_n^*\|)$.*

Proof. Let k be maximal with $m_k \neq 0$. Then $y = m_k x_k + u$ with $u \in \langle x_1, \ldots, x_{k-1} \rangle_L = \langle x_1^*, \ldots, x_{k-1}^* \rangle_L$. But $x_k = x_k^* + v$, with v also lying in $\langle x_1^*, \ldots, x_{k-1}^* \rangle_L$. It follows that $y = m_k x_k^* + \sum_{i=1}^{k-1} r_i x_i^*$, where $r_i \in \mathbb{R}$. So since the x_i^* are orthogonal, $\|y\|^2 = m_k^2 \|x_k^*\|^2 + \sum_{i=1}^{k-1} r_i^2 \|x_i^*\|^2 \geq \|x_k^*\|^2$. □

4.3 Reduced lattice bases

Throughout this section we fix $\alpha \in \mathbb{R}$ with $\frac{1}{4} < \alpha < 1$ and set $\beta = \frac{4}{4\alpha - 1}$, so $\frac{4}{3} < \beta < \infty$.

Definition 4.3.1 *Let $x_1, \ldots, x_n \in \mathbb{R}^n$ be a basis of a lattice L, with Gram-Schmidt orthogonalisation $x_1^*, \ldots, x_n^*$, and corresponding GS-coefficients μ_{ji}. This basis is called α-reduced if*

1. *$|\mu_{ji}| \leq \frac{1}{2}$, for $1 \leq i < j \leq n$,*
2. *$\|x_i^* + \mu_{i,i-1} x_{i-1}^*\|^2 \geq \alpha \|x_{i-1}^*\|^2$ for $2 \leq i \leq n$.*

Concerning the second condition compare Lemma 4.2.4. By interchanging x_{i-1}, x_i we get a new $\hat{x}_{i-1}^*$, and the second condition says that the length of the new $\hat{x}_{i-1}^*$ is not much smaller than the length of the old x_{i-1}^*

The next proposition says that the first vector in an α-reduced basis is "quite short", and it also quantifies this notion.

Proposition 4.3.2 *Let $x_1, \ldots, x_n$ be an α-reduced basis of a lattice $L \subset \mathbb{R}^n$. Let y be any nonzero element of L. Then $\|x_1\| \leq \beta^{\frac{n-1}{2}} \|y\|$.*

Proof. Using first the second and then the first condition of Definition 4.3.1 we get

$$\|x_i^*\|^2 \geq (\alpha - \mu_{i,i-1}^2)\|x_{i-1}^*\|^2 \geq (\alpha - \frac{1}{4})\|x_{i-1}^*\|^2 = \frac{1}{\beta}\|x_{i-1}^*\|^2.$$

But $x_1^* = x_1$, so this yields $\|x_1\|^2 = \|x_1^*\|^2 \leq \beta \|x_2^*\|^2 \leq \beta^2 \|x_3^*\|^2 \leq \cdots \leq \beta^{n-1}\|x_n^*\|^2$. So $\|x_i^*\|^2 \geq \beta^{-i+1}\|x_1\|^2$. Finally, by Lemma 4.2.5 we see $\|y\| \geq \min(\|x_1^*\|, \ldots, \|x_n^*\|) \geq \beta^{\frac{-n+1}{2}} \|x_1\|$. □

Next we prove an extension of this proposition, bounding the norm of all elements of an α-reduced basis. For this we first need a lemma.

Lemma 4.3.3 *Let $x_1, \ldots, x_n$ be an α-reduced basis of a lattice $L \subset \mathbb{R}^n$. Then $\|x_i\|^2 \leq \beta^{j-1}\|x_j^*\|^2$, for $1 \leq i \leq j \leq n$.*

Proof. As seen in the proof of Proposition 4.3.2 we have $\|x_{j-1}^*\|^2 \leq \beta \|x_j^*\|^2$. So $\|x_i^*\|^2 \leq \beta^{j-i}\|x_j^*\|^2$.

From (4.1) we get $x_j = x_j^* + \sum_{i=1}^{j-1} \mu_{ji} x_i^*$. So, since the x_i^* are orthogonal: $\|x_j\|^2 = \|x_j^*\|^2 + \sum_{i=1}^{j-1} \mu_{ji}^2 \|x_i^*\|^2$. Putting these things together, and using the first condition of Definition 4.3.1, we get

$$\|x_j\|^2 \leq \|x_j^*\|^2 + \sum_{i=1}^{j-1} \frac{1}{4}\beta^{j-i}\|x_j^*\|^2 = \left(1 + \frac{1}{4}\sum_{i=1}^{j-1}\beta^{j-i}\right)\|x_j^*\|^2 = \left(1 + \frac{1}{4}\frac{\beta^j - \beta}{\beta - 1}\right)\|x_j^*\|^2.$$

Now we claim that $(1 + \frac{1}{4}\frac{\beta^j - \beta}{\beta - 1}) \leq \beta^{j-1}$. Suppose that this claim is proved. Then $\|x_j\|^2 \leq \beta^{j-1}\|x_j^*\|^2$, whence $\|x_i\|^2 \leq \beta^{i-1}\|x_i^*\|^2 \leq \beta^{j-1}\|x_j^*\|^2$.

In order to prove the claim, write $u(j) = 1 + \frac{1}{4}\frac{\beta^j - \beta}{\beta - 1}$. We proceed by induction on j. We have $u(1) = 1 \leq 1 = \beta^0$. For the induction step, suppose that $u(j) \leq \beta^{j-1}$, for some $j \geq 1$.

Then $u(j+1) \leq \beta u(j)$: indeed, after multiplying both sides by $4(\beta-1)$ (which is positive), and some manipulation, we get that this is equivalent to $(\beta-1)(3\beta-4) \geq 0$, which holds. So $u(j+1) \leq \beta u(j) \leq \beta^j$. □

Theorem 4.3.4 *Let $x_1, \ldots, x_n$ be an α-reduced basis of a lattice $L \subset \mathbb{R}^n$. Let $m \leq n$ and let $y_1, \ldots, y_m \in L$ be linearly independent. Then for $1 \leq i \leq m$:*

$$\|x_i\| \leq \beta^{\frac{n-1}{2}} \max\{\|y_1\|, \ldots, \|y_m\|\}.$$

Proof. Write $y_i = \sum_{j=1}^{n} r_{ij} x_j$, $r_{ij} \in \mathbb{Z}$. For $1 \leq i \leq m$ let k_i denote the largest index j for which $r_{ij} \neq 0$. Then

$$y_i = \sum_{j=1}^{k_i} r_{ij} x_j = \sum_{j=1}^{k_i} r_{ij} \sum_{l=1}^{j} \mu_{jl} x_l^* = \sum_{j=1}^{k_i} \sum_{l=1}^{j} r_{ij} \mu_{jl} x_l^*.$$

This is equal to $r_{i,k_i} x_{k_i}^* + \nu_{i,k_i-1} x_{k_i-1}^* + \cdots + \nu_{i,1} x_1^*$, with $\nu_{i,l} \in \mathbb{Q}$. So since r_{i,k_i} is a nonzero integer we see that $\|y_i\|^2 \geq \|x_{k_i}^*\|^2$.

After possibly reordering the y_i we may assume that $k_1 \leq k_2 \leq \cdots \leq k_m$. Suppose that $k_i < i$ for some i. Then $y_1, \ldots, y_i$ all lie in the span of $x_1, \ldots, x_{i-1}$. But that means that they cannot be linearly independent. Hence $k_i \geq i$ for all i. So we can take $j = k_i$ in Lemma 4.3.3, and get

$$\|x_i\|^2 \leq \beta^{k_i-1} \|x_{k_i}^*\|^2 \leq \beta^{n-1} \|x_{k_i}^*\|^2 \leq \beta^{n-1} \|y_i\|^2.$$

So since $\|y_i\|^2 \leq \max\{\|y_1\|^2, \ldots, \|y_m\|^2\}$, this completes the proof. □

4.4 The LLL algorithm

Now we turn to an algorithm for computing an α-reduced basis of a lattice $L \subset \mathbb{R}^n$. The algorithm works with several related objects, which together we call the *state* of the algorithm. In detail, a state of the algorithm is a quadruple $S = (X, X^*, M, \gamma^*)$, where

- X is an $n \times n$-matrix whose rows, denoted $x_1, \ldots, x_n$, form a basis of L,
- the rows of X^* are the elements of the Gram-Schmidt orthogonalisation, $x_1^*, \ldots, x_n^*$,
- the matrix M contains the GS-coefficients, so $M(j,i) = \mu_{ji}$,
- the vector $\gamma^* = (\gamma_1^*, \ldots, \gamma_n^*)$ contains the squared norms of the x_i^*, i.e., $\gamma_i^* = (x_i^*, x_i^*)$.

It is clear that the first component, X, determines all other components. If in the algorithm we write, for example, x_i^* then this is meant relative to the current state.

We recall the definition of the k-th Gram determinant, $d_k = \det(G_k)$, where $G_k = X_k X_k^T$.

In the algorithm the state is changed by means of two basic operations: **Reduce**(k,l) and **Exchange**(k). First we briefly describe them.

4.4.1 **Reduce**(k, l)

Here we have $k > l$, and

- if $|\mu_{k,l}| \leq \frac{1}{2}$ then this procedure does nothing;
- otherwise we let ν be the integer closest to $\mu_{k,l}$ (this is defined as $\nu = \lceil \mu_{k,l} - \frac{1}{2} \rceil$, so if $\mu_{k,l} = m + \varepsilon$, with $0 \leq \varepsilon < 1$ then $\nu = m$ if $0 \leq \varepsilon \leq \frac{1}{2}$ and $\nu = m + 1$ otherwise), and replace x_k by $x_k - \nu x_l$, and all other x_i are left unchanged.

Lemma 4.4.1 *Let (X, X^*, M, γ^*) be a state and apply* **Reduce**(k, l)*, where $k > l$. Let (Y, Y^*, N, δ^*) be the state afterwards. Write $\nu_{j,i} = N(j, i)$. Define the $n \times n$-matrix E by $E(i, i) = 1$, $1 \leq i \leq n$, $E(k, l) = -\nu$, and all other entries are 0. Then*

1. *$Y = EX$, $N = EM$, $Y^* = X^*$ and $\delta^* = \gamma^*$,*
2. *$\nu_{j,i} = \mu_{j,i}$ if $j \neq k$, or $j = k$ and $i > l$, and $|\nu_{k,l}| \leq \frac{1}{2}$,*
3. *write $d_i = \det(X_i X_i^T)$, $d_i' = \det(Y_i Y_i^T)$, then $d_i = d_i'$ for all i.*

Proof. Let A be the $n \times n$-matrix with rows $a_1, \ldots, a_n$. Then the k-th row of EA is $a_k - \nu a_l$, and the i-th row ($i \neq k$) of EA is a_i. This implies $Y = EX$.

Since $l < k$ we have that $\langle x_1, \ldots, x_i \rangle_L = \langle y_1, \ldots, y_i \rangle_L$ for all i. So, as x_i^* is the projection of x_i onto the orthogonal complement of $\langle x_1, \ldots, x_{i-1} \rangle_L$ we get $y_i^* = x_i^*$ for all i. In other words, $Y^* = X^*$. This also implies $(EM)X^* = E(MX^*) = EX = Y = NY^* = NX^*$, and since X^* is invertible, $N = EM$.

For the second statement, note that, if $j \neq k$, then the j-th row of N is equal to the j-th row of M, whereas the k-th row of N is equal to the k-th row of M minus ν times the l-th row of M, i.e., $\nu_{k,i} = \mu_{k,i} - \nu \mu_{l,i}$. But if $i > l$ then $\mu_{l,i} = 0$, whence $\nu_{k,i} = \mu_{k,i}$. Furthermore, $\nu_{k,l} = \mu_{k,l} - \nu \mu_{l,l} = \mu_{k,l} - \nu$, as $\mu_{l,l} = 1$. By the definition of ν it now follows that $|\nu_{k,l}| \leq \frac{1}{2}$.

By Lemma 4.2.3, $d_i = \|x_1^*\|^2 \cdots \|x_i^*\|^2 = \|y_1^*\|^2 \cdots \|y_i^*\|^2 = d_i'$. □

We conclude that updating the state after a call to **Reduce**(k, l) is straightforward. Moreover, given a lattice L with basis $x_1, \ldots, x_n$, it is straightforward to obtain a new basis of L satisfying the first condition of Definition 4.3.1. Indeed, we do the following

1. Let X be the matrix with rows $x_1, \ldots, x_n$, and compute the corresponding state.
2. For $k = 2, \ldots, n$ and $l = k - 1, k - 2, \ldots, 1$ do **Reduce**(k, l).

We call this procedure **FixCondition1**. Note that the resulting basis does not necessarily satisfy the second condition of Definition 4.3.1.

4.4.2 **Exchange**(k)

Here we assume $k > 1$; this just swaps x_{k-1} and x_k.

Lemma 4.4.2 *Let (X, X^*, M, γ^*) be a state such that $\gamma_k^* < (\alpha - \mu_{k,k-1}^2)\gamma_{k-1}^*$, where $k > 1$, and apply* **Exchange**(k)*. Let (Y, Y^*, N, δ^*) be the state afterwards. Then*

1. *$y_i^* = x_i^*$ for $i \neq k - 1, k$,*

2. $\|y_{k-1}^*\|^2 < \alpha\|x_{k-1}^*\|^2$,

3. $\|y_k^*\| \leq \|x_{k-1}^*\|$,

4. *write* $d_i = \det(X_i X_i^T)$, $d_i' = \det(Y_i Y_i^T)$, *then* $d_i = d_i'$ *for all* i, *except* $i = k-1$ *and* $d_{k-1}' \leq \alpha d_{k-1}$.

PROOF. The first statement is contained in Lemma 4.2.4. By the same lemma, $y_{k-1}^* = x_k^* + \mu_{k,k-1} x_{k-1}^*$. So $\|y_{k-1}^*\|^2 = \|x_k^*\|^2 + \mu_{k,k-1}^2 \|x_{k-1}^*\|^2 = \gamma_k^* + \mu_{k,k-1}^2 \gamma_{k-1}^* < \alpha\gamma_{k-1}^* = \alpha\|x_{k-1}^*\|^2$.

Write $U = \langle x_1, \ldots, x_{k-2}\rangle_L$, and $V = \langle x_1, \ldots, x_{k-2}, x_k\rangle_L$. By definition of the Gram-Schmidt orthogonalisation we have $x_{k-1} = x_{k-1}^* + u$ where $u \in U$ and $y_k = y_k^* + v$ where $v \in V$. But $y_k = x_{k-1}$ and hence $y_k^* = x_{k-1} - v = x_{k-1}^* + u - v$. Since $U \subset V$ we have $u - v \in V$. So because $y_k^* \in V^\perp$ we obtain $\|x_{k-1}^*\|^2 = \|y_k^* + v - u\|^2 = \|y_k^*\|^2 + \|v-u\|^2 \geq \|y_k^*\|^2$.

If $i < k-1$ then $X_i = Y_i$, so $d_i = d_i'$. Let $i > k-1$. Then Y_i is obtained from X_i by exchanging the rows $k-1$ and k. Let π be the permutation $(k-1, k)$. Write $X_i X_i^T = A$ and $Y_i Y_i^T = B$. Then $B(s,t) = A(\pi(s), \pi(t))$. Now using the Leibniz formula for determinants we get

$$\begin{aligned}
\det B &= \sum_{\sigma \in S_i} \varepsilon(\sigma) \prod_{s=1}^{i} B(s, \sigma(s)) \\
&= \sum_{\sigma \in S_i} \varepsilon(\sigma) \prod_{s=1}^{i} A(\pi(s), \pi\sigma(s)) \\
&= \sum_{\sigma \in S_i} \varepsilon(\pi^{-1}\sigma\pi) \prod_{s=1}^{i} A(\pi(s), \sigma\pi(s)) \\
&= \sum_{\sigma \in S_i} \varepsilon(\sigma) \prod_{s=1}^{i} A(s, \sigma(s)) = \det(A).
\end{aligned}$$

The statement for d_{k-1}' follows from 1. and 2. together with Lemma 4.2.3. □

We also see that updating the state after an application of **Exchange**(k) is not so straightforward. Of course, it can be done by simply recomputing all the data from Y. But it is also obvious that many things can be filled in directly. See Exercise 1 for the details.

From this lemma we get a simple algorithm to compute an α-reduced basis of a lattice given by a basis of elements in $\mathbb{Z}^n$. It consists of the following steps:

1. Perform **FixCondition1**.

2. *If* there is a k with $\|x_k^* + \mu_{k,k-1} x_{k-1}^*\|^2 < \alpha\|x_{k-1}^*\|^2$ *then* do **Exchange**(k) and return to 1. *If* there is no such k then return the obtained basis.

By Lemma 4.4.2 the product of the Gram determinants decreases every time we execute the second step. Since this product is a nonnegative integer it follows that the second step cannot be executed an infinite number of times. Hence the algorithm terminates and it is obvious that the resulting basis is α-reduced.

Now we describe a more efficient algorithm for obtaining a reduced basis of a given lattice. *Here we always assume that the lattice is given by a basis with vectors in* $\mathbb{Z}^n$. It is possible

to formulate a version of the algorithm working with basis vectors in $\mathbb{R}^n$, but here we do not go into that.

The idea for the algorithm for computing an α-reduced basis is now as follows. We say that a basis $x_1, \ldots, x_n$ has property $P(k)$ if

$$|\mu_{ji}| \leq \tfrac{1}{2}, \text{ for } 1 \leq i < j \leq k, \text{ and } \|x_i^* + \mu_{i,i-1}x_{i-1}^*\|^2 \geq \alpha \|x_{i-1}^*\|^2 \text{ for } 2 \leq i \leq k.$$

Note that any basis has $P(1)$. Suppose we have a basis with $P(k-1)$. Then we could check whether $\|x_k^* + \mu_{k,k-1}x_{k-1}^*\|^2 \geq \alpha\|x_{k-1}^*\|^2$, and if that holds we do **Reduce**(k,l) for $l = k-1, k-2, \ldots, 1$. (As seen in Lemma 4.4.1, this ensures that $|\mu_{k,l}| \leq \frac{1}{2}$.) If the condition does not hold then we perform **Exchange**(k): that does not necessarily fix a condition, but it decreases the product of the Gram determinants, so we cannot run into this case infinitely often. There is only one problem with that: **Reduce**$(k, k-1)$ may destroy the condition $\|x_k^* + \mu_{k,k-1}x_{k-1}^*\|^2 \geq \alpha\|x_{k-1}^*\|^2$ because it potentially changes $\mu_{k,k-1}$ (note that **Reduce**) does not change the x_i^*). The trick now is to **first** do **Reduce**$(k, k-1)$, and **then** check the condition.

The algorithm for computing an α-reduced basis reads as follows.

Algorithm 4.4.3
Given: a basis $x_1, \ldots, x_n \in \mathbb{Z}^n$ of the lattice $L \subset \mathbb{R}^n$.
We compute an α-reduced basis of L.

1. *Let X be the matrix with rows $x_1, \ldots, x_n$, and compute the corresponding state.*
2. *$k := 2$.*
3. *while $k \leq n$ do:*
   (a) **Reduce**$(k, k-1)$.
   (b) *If $\gamma_k^* \geq (\alpha - \mu_{k,k-1}^2)\gamma_{k-1}^*$ then*
      i. *for $l = k-2, k-3, \ldots, 1$ do* **Reduce**(k,l),
      ii. *$k := k+1$.*
   else
      i. **Exchange**(k),
      ii. *if $k > 2$ then $k := k-1$.*

Theorem 4.4.4 *Suppose that the input basis vectors $x_1, \ldots, x_n$ have integer coordinates. Then Algorithm 4.4.3 terminates, and upon termination the state contains an α-reduced basis.*

Proof. Let $D = d_1 \cdots d_{n-1}$ be the product of the first $n-1$ Gram determinants. By Lemma 4.2.3 this is a positive integer. However, after every call to **Exchange**(k) it decreases. It follows that **Exchange**(k) is called a finite number of times. This implies that the algorithm terminates, because every time the other part of the loop is entered, k is increased.

We claim that at the beginning of the loop in Step 3, when $k = k_0$ is considered, $P(k_0-1)$ holds. This is certainly true initially as then $k = 2$. A call to **Reduce**(k_0, l), does not change anything with respect to $P(k_0-1)$. So if the first clause of the if statement is entered, then afterwards $P(k_0-1)$ still holds, but also $\gamma_{k_0}^* \geq (\alpha - \mu_{k_0,k_0-1}^2)\gamma_{k_0-1}^*$, (which is the same as

$\|x^*_{k_0} + \mu_{k_0,k_0-1}x^*_{k_0-1}\|^2 \geq \alpha\|x^*_{k_0-1}\|^2$ as the x^*_i are orthogonal), and $\mu_{k_0,l} \leq \frac{1}{2}$ for $1 \leq l < k_0$. We conclude that in that case, $P(k_0)$ holds.

If the second clause of the if statement is entered, then x_{k_0} and x_{k_0-1} are interchanged, and k gets the value $k_0 - 1$. Since nothing has happened with $x_1, \ldots, x_{k_0-2}$, it follows that $P(k_0 - 2)$ holds, which is the same as $P(k - 1)$.

The conclusion is that upon termination $P(n)$ holds, so the basis is α-reduced. □

4.5 The knapsack cryptosystem

In 1978 Merkle and Hellman proposed the use of the so-called knapsack problem as the basis of a public key cryptosystem (Hiding information and signatures in trapdoor knapsacks, *IEEE Trans. Inf. Th.*, 525-530). However in 1985, Lagarias and Odlyzko showed how to break this system using the LLL algorithm.

The *knapsack problem* amounts to the following. Let $A = \{a_1, \ldots, a_n\}$ be a set of n distinct positive integers, and $s > 0$ also an integer. The problem is to decide whether there is an $I \subset \{1, \ldots, n\}$ such that

$$\sum_{i \in I} a_i = s.$$

(We can think of the a_i as the sizes of different objects to be put in a knapsack of size s; the question is whether we can exactly fill our knapsack.) An equivalent way of formulating the problem is to ask whether there is a vector $(x_1, \ldots, x_n)$, with $x_i \in \{0, 1\}$, such that $\sum_{i=1}^n x_i a_i = s$.

It is known that knapsack problems are very hard to solve in general. However, in a special case this is easy. The sequence $a_1, \ldots, a_n$ is said to be *superincreasing* if $a_j > \sum_{i=1}^{j-1} a_i$, i.e., if the j-th term is strictly bigger than the sum of the preceding terms. If our sequence is superincreasing then solving the knapsack problem is easy. First of all, $x_n = 1$ if and only if $s \geq a_n$. Secondly, once having determined x_n, we continue with the knapsack problem corresponding to the sequence $a_1, \ldots, a_{n-1}$ and $s - x_n a_n$. If we terminate in 0 then there is a solution (and we found it); otherwise there is no solution.

Example 4.5.1 Let $n = 4$ and $b_1 = 15$, $b_2 = 37$, $b_3 = 119$, $b_4 = 253$, $s = 387$. Then $x_4 = 1$ and we continue with b_1, b_2, b_3 and 134. We get $x_3 = 1$ and continue with b_1, b_2 and 15. Then $x_2 = 0$ and we finish with b_1 and 15, so that $x_1 = 1$, and we found a solution.

The idea of the knapsack cryptosystem is to work with a superincreasing sequence as the private key, so that decrypting is easy, and to give a messed-up form of this key as the public key. More specifically, one chooses a superincreasing sequence $b_1, \ldots, b_n$ such that b_1 is close to 2^n and b_n is close to 2^{2n}. Example 4.5.1 has an instance of this, with $n = 4$. Next positive integers m (the modulus) and w (the multiplier) are chosen such that

$$m > \sum_{i=1}^{n} b_i, \quad 0 < w < m, \quad \gcd(m, w) = 1,$$

the last condition implying that w is invertible modulo m. Finally a permutation π of $1, \ldots, n$ is chosen. Then the *private key*, kept by the receiver of the messages, is formed by the

sequence $b_1, \ldots, b_n$, together with m, w, and π. The *public key*, published by the receiver of the messages, is the sequence $a_1, \ldots, a_n$, where

$$a_i = wb_{\pi(i)} \bmod m$$

(where the a_i are chosen, modulo m, in the interval $(0, m)$; note that no a_i is 0, as otherwise $b_i = 0$ as well, since w is invertible modulo m).

In order to send a string $x_1 \cdots x_n$, with $x_i \in \{0, 1\}$ to the receiver, the sender computes the number $s = \sum_{i=1}^{n} x_i a_i$, and sends s. The reciever, in order to decrypt, first computes the number $t = w^{-1}s \bmod m$. Now modulo m we have the following

$$t = w^{-1}s = \sum_{i=1}^{n} x_i w^{-1} a_i = \sum_{i=1}^{n} x_i b_{\pi(i)} = \sum_{i=1}^{n} x_{\pi^{-1}(i)} b_i,$$

so the receiver has to solve a knapsack problem with a superincreasing sequence.

Example 4.5.2 Consider the b_i from Example 4.5.1 and choose $m = 451$. Furthermore, let $w = 63$. Then the $wb_i \bmod m$ are 43, 76, 281, 154. But that still is superincreasing! So this w is not good. Let's try with $w = 409$, then we get 272, 250, 414, 198, and that is not superincreasing. We also select $\pi = (1, 3, 2, 4)$, so that

$$a_1 = 414, \quad a_2 = 198, \quad a_3 = 250, \quad a_4 = 272.$$

Now suppose that the sender wants to send 1101. This is encrypted as $a_1 + a_2 + a_4 = 884$, which is sent to the receiver. We have $w^{-1} = 204 \bmod m$, so the receiver computes

$$t = 204 \cdot s \bmod m = 387.$$

Now the receiver has to solve the knapsack problem

$$x_4 b_1 + x_3 b_2 + x_1 b_3 + x_2 b_4 = 387.$$

(Note that $\pi^{-1} = (1, 4, 2, 3)$.) It is easily seen that the solution is $x_1 \cdots x_4 = 1101$ (see Example 4.5.1).

However, in order to arrive at the message that was sent one can also try the following. Consider the matrix

$$Y = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ 0 & 0 & 1 & \cdots & 0 & -a_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -a_n \\ 0 & 0 & 0 & \cdots & 0 & s \end{pmatrix}.$$

Denote its rows by $y_1, \ldots, y_{n+1}$ and let L be the lattice in $\mathbb{R}^{n+1}$ spanned by them. Note that the $\|y_i\|$ are big, but also that L contains the vector $(x_1, \ldots, x_n, 0)$ which has a much smaller length than the y_i. Now the point is that an LLL-reduced basis of L has a good chance to contain this vector, thus making the system unsafe to use.

Example 4.5.3 We continue with Example 4.5.2. Here we have

$$Y = \begin{pmatrix} 1 & 0 & 0 & 0 & -414 \\ 0 & 1 & 0 & 0 & -198 \\ 0 & 0 & 1 & 0 & -250 \\ 0 & 0 & 0 & 1 & -272 \\ 0 & 0 & 0 & 0 & 884 \end{pmatrix}.$$

In this case an LLL-reduced basis (with $\alpha = \frac{3}{4}$) is

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ -2 & 1 & -1 & 0 & -4 \\ 3 & -1 & -2 & -2 & 0 \\ -1 & 3 & -1 & -3 & 2 \\ 2 & 2 & 3 & -4 & -2 \end{pmatrix}.$$

We see that we get the decrypted form from the first row.

4.6 Exercises

1. Let (X, X^*, M, γ^*) be a state and apply **Exchange**(k). Let (Y, Y^*, N, δ^*) be the state afterwards. Write $\nu_{i,j} = N(i,j)$. The purpose of this exercise is to find formulae for the $\nu_{i,j}$.

   (a) Find a, b, c, d such that

   $$\begin{pmatrix} y^*_{k-1} \\ y^*_k \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x^*_{k-1} \\ x^*_k \end{pmatrix}.$$

   (b) Show that $ad - bc = -1$, and by inverting the matrix, that

   $$x^*_{k-1} = \mu_{k,k-1} \frac{\|x^*_{k-1}\|^2}{\|y^*_{k-1}\|^2} y^*_{k-1} + y^*_k$$
   $$x^*_k = \frac{\|x^*_k\|^2}{\|y^*_{k-1}\|^2} y^*_{k-1} - \mu_{k,k-1} y^*_k.$$

   (c) By writing y_i as linear combination of the y^*_j show that:

   i. $\nu_{j,i} = \mu_{j,i}$, for $1 \le i < j \le k-2$,

   ii. $\nu_{k-1,i} = \mu_{k,i}$, for $1 \le i \le k-2$,

   iii. $\nu_{k,i} = \mu_{k-1,i}$, for $1 \le i \le k-2$, and $\nu_{k,k-1} = \mu_{k,k-1} \frac{\|x^*_{k-1}\|^2}{\|y^*_{k-1}\|^2}$,

   iv. $\nu_{m,i} = \mu_{m,i}$ for $1 \le i \le m-1$, $i \ne k-1, k$ and $m \ge k+1$,

   v.
   $$\nu_{m,k-1} = \mu_{m,k-1}\mu_{k,k-1} \frac{\|x^*_{k-1}\|^2}{\|y^*_{k-1}\|^2} + \mu_{m,k} \frac{\|x^*_k\|^2}{\|y^*_{k-1}\|^2}$$

   vi. $\nu_{m,k} = \mu_{m,k-1} - \mu_{m,k}\mu_{k,k-1}$, $m \ge k+1$.

2. In this exercise we derive a bound for the number of times the LLL algorithm (with parameter α) executes the body of the loop in Step 3. For this, consider a state (X, X^*, M, γ^*); and let B be the maximum of the norms $\|x_i^*\|$, $1 \le i \le n$, and D the product $d_1 \cdots d_{n-1}$ (see Section 4.2 for the definition of d_i). Write B_0, D_0 for their values at the start of the algorithm.

   (a) Show that $d_k^0 \le B_0^{2k}$, where d_k^0 is the k-th Gram determinant at the start of the algorithm.

   (b) Show that $D_0 \le B_0^{n(n-1)}$.

   (c) Let N_2 be the number of times the algorithm executes the *second* clause of the *if*-statement in Step 3. Show that $\alpha^{N_2} D_0 \ge 1$, and

   $$N_2 \le -\frac{\log B_0}{\log \alpha} n(n-1).$$

   (Note that a ratio of logarithms does not depend on the base of the logarithm.)

   (d) Let S_2 be the total number of times the algorithm executes the second clause of the *if*-statement *with at the start* $k > 2$ (so this is the total number of times the statement $k := k - 1$ is executed). Let N_1 be the number of times the algorithm executes the *first* clause of the *if*-statement in Step 3. Show that $N_1 = S_2 + n - 1$.

   (e) Prove that

   $$N_1 + N_2 \le -\frac{2 \log B_0}{\log \alpha} n(n-1) + n - 1.$$

   (Note that $N_1 + N_2$ is the total number of rounds of the iteration.)

3. Let $\alpha = \frac{3}{4}$ and $x_1 = (4, -3)$, $x_2 = (3, -3)$. Compute an α-reduced basis of the lattice in $\mathbb{R}^2$ spanned by x_1, x_2.