Computing an arithmetic subgroup of a unipotent algebraic group

Andrea Pavan

Suppose we are given an affine algebraic group $\mathbf{G}$ over $\mathbb{Q}$ acting faithfully on a finite dimensional vector space V, together with a full-dimensional lattice of V. Then we can consider the group $\mathbf{G}(\mathbb{Q})$ of the rational points of $\mathbf{G}$, as well as its subgroup $\mathbf{G}_L$ consisting of the points stabilizing the lattice L. $\mathbf{G}_L$ and, more generally, any subgroup of $\mathbf{G}(\mathbb{Q})$ commensurable with it, is a so-called arithmetic subgroup of $\mathbf{G}$. It can be shown that this is a good definition, in the sense that the class of arithmetic subgroups of $\mathbf{G}$ does not depend on the choice of the action (as long as it is faithful) nor on the choice of the lattice (as long as it is full-dimensional). Further, Borel and Harish-Chandra proved that any arithmetic subgroup of an affine algebraic group over $\mathbb{Q}$ is finitely presented. Later on, Grunewald and Segal provided an effective method to compute a generating system of a given arithmetic subgroup. Unfortunately, their algorithm seems to be quite expensive from a computational point of view, hence not well-suited for real-world applications. Also, the computed set of generators does not have in general nice algebraic properties.

In this talk we address the problem in the special case in which $\mathbf{G}$ is unipotent. Under this hypothesis, the action of $\mathbf{G}$ on V has a very special form. Also, the usual correspondence between algebraic groups and Lie algebras becomes stronger if we restrict to unipotent groups (and nilpotent algebras). We will explain how to exploit these two facts in order to provide a new algorithm capable to compute a generating system for $\mathbf{G}_L$, and with two enjoyable extra properties. From one side, its computational cost is fairly low, hence it can be effectively employed in many interesting cases. On the other side, the computed set of generators exhibits a normal central series of $\mathbf{G}_L$. Finally, we will show how the algorithm can be slightly modified in order to compute even a finite presentation of $\mathbf{G}_L$; as a byproduct this leads to an indipendent proof of the theorem of Borel and Harish-Chandra for the unipotent case.