Polynomial functions on finite point sets

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Let $\mathbb{F}$ be a field. We consider polynomial functions on finite point sets $\mathcal{V} \subset \mathbb{F}^n$. The polynomial functions mapping $\mathcal{V}$ to $\mathbb{F}$ carry a wealth of information about the combinatorial and geometric properties of $\mathcal{V}$. To obtain information about these functions, it is useful to consider the ideal $I(\mathcal{V})$:

$$I(\mathcal{V}) := \{f \in \mathbb{F}[x_1, \dots, x_n] : \ f(v) = 0 \text{ whenever } v \in \mathcal{V}\}.$$

Our principal aim is to describe Gröbner bases and related structures (primarily standard monomials and Hilbert functions) for some combinatorially significant sets $\mathcal{V}$. We will outline applications to combinatorics to demonstrate the effectiveness of the approach.

Let $\mathbb{F}$ be a field, and n be a positive integer. Let $v_F \in \{0,1\}^n$ denote the characteristic vector of a set $F \subseteq [n]$. For a family of subsets $\mathcal{F} \subseteq 2^{[n]}$, let $V(\mathcal{F}) = \{v_F : F \in \mathcal{F}\} \subseteq \{0,1\}^n \subseteq \mathbb{F}^n$.

We give explicit descriptions of the Gröbner bases and normal sets of $I(V(\mathcal{F}))$ for several interesting set families $\mathcal{F} \subseteq 2^{[n]}$, including the complete uniform families $\mathcal{F} = \binom{[n]}{d}$, and some generalizations of them. Also, we determine the lex and deglex standard monomials (normal set) of point sets associated to partitions. Applications related to inclusion matrices in combinatorics will be outlined. For example, we present a generalization of Wilson's rank formula for incidence matrices.

We describe a general method, called *lex game*, for computing the lex standard monomials of finite general points sets $\mathcal{V}$. This two player combinatorial game allows us to describe the normal set of some important symmetric set families with the aid of lattice paths. The method also leads to a fast algorithm for the computation of the lex normal set of $\mathcal{V}$.