# Constructing Faithful Matrix Representations of Lie Algebras

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**Abstract**

By Ado's theorem every finite dimensional Lie algebra over a field of characteristic zero has a faithful finite dimensional representation. We consider the algorithmic problem of constructing such a representation for Lie algebras given by a multiplication table. An effective version of Ado's theorem is proved.

## 1 Introduction

When dealing with the problem of representing finite dimensional Lie algebras on computer, two presentations leap into mind: a presentation by matrices and a presentation by an array of structure constants. In the first presentation the Lie algebra is given by a finite set of matrices $\{A_1, \dots, A_n\}$ that form a basis of the Lie algebra. If $A$ and $B$ are two elements of the space spanned by the $A_i$, then their Lie product is defined as $[A, B] = A \cdot B - B \cdot A$ (where the $\cdot$ denotes the ordinary matrix multiplication). The second approach is more abstract. The Lie algebra is a (abstract) vector space over a field $F$ with basis $\{x_1, \dots, x_n\}$ and the Lie multiplication is determined by

$$[x_i, x_j] = \sum_{k=1}^{n} c_{ij}^k x_k.$$

Here $(c_{ij}^k) \in F^3$ is an array of $n^3$ structure constants that determines the Lie multiplication completely.

By linear algebra it is seen that from a presentation of a Lie algebra by matrices, we easily can obtain a presentation of it by structure constants. Ado's theorem states that the opposite direction is also possible.

**Theorem 1.1 (Ado)** *Let $L$ be a Lie algebra over a field of characteristic 0. Then $L$ has a faithful finite dimensional representation.*

However, the standard proofs of this theorem (see [3], [5]) do not provide effective constructions. The problem we are dealing with here is that of effectively constructing a faithful representation by matrices for a Lie algebra given by structure constants.

A partial solution is provided by the *adjoint representation*,

$$\mathrm{ad} : L \longrightarrow \mathfrak{gl}(L),$$

defined by $\mathrm{ad}(x)(y) = [x, y]$. The fact that this is a Lie algebra representation is equivalent to the Jacobi identity. We consider the kernel of this representation. Suppose $\mathrm{ad}(x)(y) = 0$ for all $y \in L$; then $[x, y] = 0$ for all $y \in L$. Hence $x \in Z(L)$, the centre of $L$. So for Lie algebras with a trivial centre the problem is solved by the adjoint representation. The rest of the paper will be concerned with Lie algebras that have a nontrivial centre.

Here we describe an effective method for constructing a faithful finite dimensional representation of a Lie algebra $L$. The method follows roughly the lines of the proof of Ado's theorem given in [3]. In this proof a tower of Lie algebra extensions (where every term is an ideal in the next one) is constructed with final term $L$. An algorithm for computing such a tower is given in Section 2. A representation of the first element of the tower is easily constructed. Then this representation is successively extended to representations of the higher terms of the tower and finally to $L$ itself. Sections 3 and 4 focus on a single extension step. In Section 3 the vector space underlying the extension is described. We take a significantly smaller space than is done in the proof in [3]. Then in Section 4 it is proved that it is sufficient to work with this smaller space. An algorithm for calculating the extension is given. In Section 5 an algorithm for the construction of a faithful finite dimensional representation of $L$ is given and Ado's theorem is obtained as a corollary. Also an upper bound on the degree of the resulting representation is given in the case where $L$ is nilpotent. Finally in Section 6, some practical examples are discussed.

## 2 Calculating a series of extensions

Here we describe how a series of subalgebras $K_1 \subset K_2 \subset \cdots \subset K_m = L$ can be constructed such that $K_{i+1} = K_i \rtimes H_i$. This means the following:

- $K_i$ is an ideal of $K_{i+1}$,
- $H_i$ is a subalgebra of $K_{i+1}$,
- $K_{i+1} = K_i \oplus H_i$ (direct sum of vector spaces).

In the sequel $\mathrm{NR}(K)$ will denote the nilradical of $K$ (see [5], §1.7). The following algorithm calculates a series as described above.

**Algorithm** `ExtensionSeries`
Input: A Lie algebra $L$ over a field of characteristic 0.
Output: Series $K_1, K_2, \dots, K_r = L$ and $H_1, \dots, H_{r-1}$ such that

1. $K_1$ is commutative,
2. $K_{i+1} = K_i \rtimes H_i$,
3. $[H_i, K_i] \subset \mathrm{NR}(K_i)$ for $1 \leq i \leq r-1$,
4. $\dim H_i = 1$ for $1 \leq i < r-1$.

$R$ :=`SolvableRadical`($L$);
$K_1$ := the final term of the derived series of $R$;
$i := 1$;
**while** $K_i \neq R$ **do**
  $I$ := the unique element of the derived series of $R$ such that $[I, I] \subset K_i$, but $I$ is not contained in $K_i$;
  $y$ := an element from $I \setminus K_i$;
  $K_{i+1} := K_i \rtimes \langle y \rangle$;
  $H_i := \langle y \rangle$;
  $i := i + 1$;
**od**;
$r := i + 1$;
$K_r := L$; $H_{r-1}$ := `LeviSubalgebra`($L$);

**Proof.** First we consider the computability of all the steps. Algorithms for the computation of the solvable radical and of the derived series are described in [1]. A Levi subalgebra of $L$ is a semisimple subalgebra $S$ such that $L = R \rtimes S$. In [4] and [6] algorithms for the computation of a Levi subalgebra are given.

In the first part of the algorithm a series of subalgebras

$$0 \subset K_1 \subset K_2 \subset \cdots \subset K_{r-1} = R$$

is constructed such that $K_{i+1} = K_i \rtimes \langle y_i \rangle$. From the choice of $y_i$ it is seen that $K_i$ is indeed an ideal in $K_{i+1}$. For $1 \leq i < r-1$ we let $H_i$ be the 1-dimensional subalgebra spanned by $y_i$. At the end we let $H_{r-1}$ be a Levi subalgebra and we set $K_r = L$.

The first two properties of the output are immediate. We have that $[L, R] \subset \mathrm{NR}(L)$ (Theorem 2.13 of [5]) and $K_i \subset R$ for $1 \leq i \leq r-1$. Hence $[H_i, K_i] \subset \mathrm{NR}(L) \cap K_i \subset \mathrm{NR}(K_i)$. The last inclusion follows from the fact that $\mathrm{ad}_{K_i} x$ is nilpotent for all $x \in \mathrm{NR}(L) \cap K_i$.

Finally from the construction above it is seen that $\dim H_i = 1$ for $1 \leq i < r-1$. $\square$

## 3 The extension space

Here we consider the situation where $L = K \rtimes H$ and we suppose that there is a finite dimensional representation $\rho : K \to \mathfrak{gl}(V)$ of $K$. We try to find a finite dimensional representation $\sigma$ of $L$. Under some conditions we succeed in doing this.

First we describe the space on which $L$ is to be represented. By $U(K)$ we will denote the universal enveloping algebra of $K$. If $\{x_1, \dots, x_t\}$ is a basis of $K$, then by the Poincaré-Birkhoff-Witt (PBW) theorem ([5], Theorem 5.3) a basis of $U(K)$ (called PBW-basis) is formed by the *standard monomials* $x_1^{k_1}\cdots x_t^{k_t}$. Hence a basis of the dual space $U(K)^*$ is formed by the functionals $f_a$, defined by $f_a(b) = 1$ if $b = a$ and it is 0 if $b \neq a$, where $a$ runs through the PBW-basis of $U(K)$.

The representation space of $L$ will be a finite dimensional subspace of $U(K)^*$. First we describe how $L$ acts on $U(K)^*$. Let $f$ be an element of $U(K)^*$ and let $x \in K$ and $y \in H$. Then for $a \in U(K)$ we set

$$\begin{aligned} (x \cdot f)(a) &= f(ax) \\ (y \cdot f)(a) &= -f(ya - ay). \end{aligned}$$

Note that for $a \in U(K)$ we have that $ya - ay$ lies also in $U(K)$. By some simple calculations it can be shown that this is indeed a Lie algebra action ([3], §7.2).

We extend the representation $\rho$ of $K$ to a representation of the universal enveloping algebra $U(K)$, by

$$\rho(x_1^{k_1}\cdots x_t^{k_t}) = \rho(x_1)^{k_1}\cdots\rho(x_t)^{k_t}.$$

Consider the map

$$\theta : V \times V^* \longrightarrow U(K)^*$$

defined by $\theta(v, w^*)(a) = w^*(\rho(a)v)$. An element $\theta(v, w^*)$ is called a *coefficient* of the representation $\rho$. By $C_\rho$ we denote the image of $\theta$ in $U(K)^*$. For the proof of the following lemma we refer to [3], §7.1.

**Lemma 3.1** *$C_\rho$ is a $K$-submodule in $U(K)^*$.*

Let $S_\rho \subset U(K)^*$ be the $L$-submodule of $U(K)^*$ generated by $C_\rho$. Let $\sigma : L \to \mathfrak{gl}(S_\rho)$ be the corresponding representation. In [3] the space $S_\rho^n$ is taken as the vector space underlying the representation. Then it is proved that the corresponding representation contains a copy of $\rho$. This is not guaranteed to hold for $\sigma$. However, by a slight abuse of language we will call $\sigma$ the *extension* of $\rho$ to $L$.

The next proposition states some conditions under which $S_\rho$ is finite dimensional.

**Proposition 3.2** *Let $L = K \rtimes H$ such that $[H, K] \subset \mathrm{NR}(K)$ and let $\rho : K \to \mathfrak{gl}(V)$ be a finite dimensional representation of $K$ such that $\rho(x)$ is nilpotent for all $x \in \mathrm{NR}(K)$. Let $\sigma : L \to \mathfrak{gl}(S_\rho)$ be the extension of $\rho$ to $L$. Then we have that $S_\rho$ is finite dimensional and $\sigma(x)$ is nilpotent for all $x \in \mathrm{NR}(L)$.*

**Proof.** The proof of these facts can be found in the proof of Theorem 1 of [3] §7.2. □

## 4 Extending a representation

Here we show how a faithful finite dimensional representation of a Lie algebra $L$ can be constructed working with the extension described in the previous section.

Throughout this section $L = K \rtimes H$ and $\rho : K \to \mathfrak{gl}(V)$ will be a finite dimensional representation of $K$. Furthermore, $\sigma : L \to \mathfrak{gl}(S_\rho)$ will be the extension of $\rho$ to $L$.

The key to the algorithm will be the following proposition.

**Proposition 4.1** *Suppose that $\rho$ is a faithful representation of $K$. Then $\sigma$ is faithful on $K$. Furthermore, if $H$ is 1-dimensional, then $\sigma$ is a faithful representation of $L$ or there is an element $\tilde{y} \in K$ such that $y - \tilde{y} \in Z(L)$, where $y$ is an element spanning $H$.*

**Proof.** Let $x$ be an element of $K$. Then $\rho(x) \neq 0$ and hence there is a $v \in V$ and a $w^* \in V^*$ such that

$$0 \neq w^*(\rho(x)v) = \sigma(x) \cdot \theta(v, w^*)(1).$$

Hence $\sigma(x) \neq 0$ for all $x \in K$ so that $\sigma$ is faithful on $K$.

Let $\{x_1, \ldots, x_t\}$ be a basis of $K$ and suppose that $H = \langle y \rangle$. Suppose further that $\sigma$ is not faithful on $L$. This means that there is a nontrivial relation

$$\lambda\sigma(y) + \sum_{i=1}^{t} \lambda_i \sigma(x_i) = 0.$$

Because $\sigma$ is faithful on $K$, we may assume that $\lambda = 1$. It follows that there is a $\tilde{y} \in K$ such that $\sigma(y) = \sigma(\tilde{y})$. Then for all $x \in K$ we have

$$\sigma([\tilde{y}, x]) = [\sigma(\tilde{y}), \sigma(x)] = [\sigma(y), \sigma(x)] = \sigma([y, x]).$$

Since $\sigma$ is faithful on $K$, this implies that $[\tilde{y}, x] = [y, x]$. Also $\sigma([y, y - \tilde{y}]) = 0$ and because $[y, y - \tilde{y}] \in K$ we have that it is 0. The conclusion is that $y - \tilde{y} \in Z(L)$. □

Now we continue with some observations about the space $S_\rho$. In the sequel $\{y_1, \ldots, y_s\}$ will be a basis of $H$, and $\{v_1, \ldots, v_n\}$ will be a basis of $V$. By $e_{ij}^n$ we denote the $n \times n$ matrix with a 1 on position $(i, j)$ and zeros elsewhere.

**Lemma 4.2** *Suppose that* $\rho(x)v_1 = 0$ *for all elements* $x \in K$*. Then there is a basis*

$$\{w_1, \ldots, w_m\}$$

*of* $S_\rho$ *such that* $\sigma(x)w_1 = 0$ *for all* $x \in L$*.*

**Proof.** We work with the customary dual basis (with respect to $v_1, \ldots, v_n$), $\{v_1^*, \ldots, v_n^*\}$ of $V^*$ (i.e., $v_i^*(v_j) = \delta_{ij}$). Set $w_1 = \theta(v_1, v_1^*)$. Let $a$ be a monomial in $U(K)$. We calculate $w_1(a) = \theta(v_1, v_1^*)(a) = v_1^*(\rho(a)v_1)$; it is 0 if $a \neq 1$ and 1 if $a = 1$. It follows that $w_1$ takes the value 1 on the element 1 of $U(K)$ and 0 on all other monomials. In particular $w_1$ is nonzero. Now we extend $w_1$ to a basis

$$w_1, \ldots, w_m$$

of $S_\rho$. Let $a$ be an element of $U(K)$. If $x$ is an element of $K$ then

$$\sigma(x)w_1(a) = w_1(ax).$$

The support of $ax$ does not contain a constant term, hence $w_1(ax) = 0$. Now let $x \in H$. Then

$$\sigma(x)w_1(a) = -w_1(xa - ax).$$

Since the support of $xa - ax$ also does not contain a constant term, we have that $w_1(xa - ax) = 0$. It follows that $\sigma(x)w_1 = 0$ for all $x \in L$. □

**Lemma 4.3** *The space* $S_\rho$ *is spanned by the elements*

$$y_1^{k_1} \cdots y_s^{k_s} \cdot \theta(v_i, v_j^*)$$

*where* $k_q \geq 0$ $(1 \leq q \leq s)$ *and* $1 \leq i, j \leq n$*.*

**Proof.** Let $\{x_1, \ldots, x_t\}$ be a basis of $K$. Then by the PBW theorem $S_\rho$ is spanned by all elements of the form

$$y_1^{k_1} \cdots y_s^{k_s} \cdot x_1^{l_1} \cdots x_t^{l_t} \cdot \theta(v_i, v_j^*).$$

But since $C_\rho$ is a $U(K)$-module (Lemma 3.1), we have that such element is a linear combination of elements of the form

$$y_1^{k_1} \cdots y_s^{k_s} \cdot \theta(v_k, v_l^*).$$

□

Let $a$ be an element of $U(K)$. By $\mathrm{Orb}_H(a)$ we denote the orbit of $a$ under the action of the elements of $H$, i.e.,

$$\mathrm{Orb}_H(a) = \langle y_1^{k_1} \cdots y_s^{k_s} \cdot a \mid k_1, \ldots, k_s \geq 0 \rangle,$$

where $y_i \cdot a = y_i a - a y_i$.

**Lemma 4.4** *Let* $f \in S_\rho$*. If* $a$ *is an element of* $U(K)$ *such that* $\rho(\mathrm{Orb}_H(a)) = 0$*, then* $f(a) = 0$*.*

**Proof.** Set $g = y_1^{k_1} \cdots y_s^{k_s} \cdot \theta(v_i, v_j^*)$, then

$$g(a) = \pm v_j^*(\rho(y_s^{k_s} \cdots y_1^{k_1} \cdot a)v_i) = 0.$$

Since $f$ is a linear combination of elements of this form (Lemma 4.3), we have that $f(a) = 0$. □

**Remark.** Let $a$ be an element of $U(K)$ of degree $d$. Let $W$ be the span of all monomials in $U(K)$ of degree $\leq d$. Then $\mathrm{Orb}_H(a) \subset W$. The conclusion is that $\mathrm{Orb}_H(a)$ is finite dimensional. By viewing it as a subspace of $W$, we can calculate a basis of $\mathrm{Orb}_H(a)$.

Now we formulate an algorithm for extending the representation $\rho$ to $L$. There are two cases to be considered; the general case and the case where $H = \langle y \rangle$ and there is a $\tilde{y} \in K$ such that $y - \tilde{y} \in Z(L)$. In the second case we can easily construct a representation of $L$. Then by Proposition 4.1 we always obtain a faithful representation of $L$ in the case where $H$ is 1-dimensional.

For greater clarity we formulate the algorithm using a subroutine that treats the general case. We first state the subroutine.

**Algorithm `GeneralExtension`**
Input: $L = K \rtimes H$ and $\rho : K \longrightarrow \mathfrak{gl}(V)$.
Output: The extension $\sigma : L \longrightarrow \mathfrak{gl}(S_\rho)$.

Step 1 Calculate a set of standard monomials $\{m_1, \ldots, m_r\}$ that form a basis of a complement to $\ker \rho$ in $U(K)$.

Step 2 Calculate a basis of $C_\rho$.

Step 3 $d := \max \deg m_i$;

Step 4 $V_d := \{a \in U(K) \mid a$ is a monomial of degree $\leq d$ such that $\rho(\mathrm{Orb}_H(a)) \neq 0\}$;

Step 5 Calculate a basis of $S_\rho$ (relative to $V_d$). And let the first basis element be $\theta(v_1, v_1^*)$.

Step 6 Calculate the action of the elements of a basis of $L$ on $S_\rho$. If this yields a representation of $L$, then return that representation. Otherwise set $d := d + 1$; and go to Step 4.

**Proof.** The algorithm is straightforward. It calculates a basis of $S_\rho$ and the matrices of the corresponding representation. A function in $S_\rho$ is described by giving its values on a finite set of (standard) monomials in $U(K)$. This enables us to represent every element of $S_\rho$ by a vector of finite length so that we can use linear algebra to calculate a basis and the coefficients of an element with respect to that basis.

Most of the steps are concerned with finding an appropriate set $V_d$ of monomials. First we consider the space $C_\rho$. We have that

$$\theta(v_i, v_j^*)(a) = v_j^*(\rho(a)v_i),$$

so that we can describe a function in $C_\rho$ by giving its values on the monomials $m_i$ constructed in Step 1. Now we let $V_d$ be a subset of the set of all monomials of degree $\leq d$. So initially we set $d$ equal to the maximum degree of a monomial $m_i$, ensuring that all these elements will be contained in $V_d$. By Lemma 4.4 we may discard all monomials $a$ such that $\rho(\mathrm{Orb}_H(a)) = 0$. Using Lemma 4.3 we calculate a basis of $S_\rho$, representing each function on the set $V_d$. Then we calculate the matrices of the action of the elements of a basis of $L$. If this yields a representation of $L$ then we are done. Otherwise we apparently did not calculate all of $S_\rho$ in the preceding step. This means that there are functions in $S_\rho$ that cannot be described by giving their values on only the monomials in $V_d$. So in this case we set $d := d+1$ and go through the process again. $\Box$

Now we state the routine that also treats the special case.

**Algorithm** `ExtendRepresentation`
Input: $L = K \rtimes H$ and $\rho : K \to \mathfrak{gl}(V)$ such that $\rho(x) \cdot v_1 = 0$ for $x \in K$.
Output: An extension $\sigma : L \to \mathfrak{gl}(W)$.

**if** $H = \langle y \rangle$ and there is a $\tilde{y} \in K$ such that
  $y - \tilde{y} \in Z(L)$
**then**
  $n := \dim V$;
  $\sigma(y - \tilde{y}) := e_{1,n+1}^{n+1}$;
  **for** $x$ in a basis of $K$ **do**
    $\sigma(x) :=$ the $n+1 \times n+1$ matrix of which the $n \times n$ submatrix in the top left corner is $\rho(x)$ and the other positions are 0;
  **od**;
**else** $\sigma :=$ `GeneralExtension`$(L, \rho)$;
**fi**;

**Proof.** First we remark that finding a $\tilde{y}$ such that $y - \tilde{y} \in Z(L)$ amounts to solving a system of linear equations.

We have to prove that the map $\sigma$ constructed in the first part of the algorithm is a representation of $L$. Since $\rho(x) \cdot v_1 = 0$ for all $x \in K$, we have that the first column of the matrix $\rho(x)$ is zero. Hence $\sigma(y - \tilde{y})$ commutes with $\rho(x)$ for $x \in K$. $\Box$

## 5 An effective version of Ado's theorem

Using the routines `ExtensionSeries` and `ExtendRepresentation`, we formulate an algorithm for calculating a finite dimensional faithful representation of an arbitrary Lie algebra of characteristic 0.

**Algorithm** `Representation`
Input: A Lie algebra $L$.
Output: A finite dimensional faithful representation $\sigma$ of $L$.

$[K_1, \ldots, K_r, H_1, \ldots, H_{r-1}] :=$ `ExtensionSeries`$(L)$;
$\rho_1(x_i) := e_{1,i+1}^{s+1}$;
*(Where $\{x_1, \ldots, x_s\}$ is a basis of $K_1$)*
$i := 2$;
**while** $i \leq r-1$ **do**
  $\rho_i :=$`ExtendRepresentation`$(\rho_{i-1}, K_i)$;
  $i := i+1$;
**od**;
**if** $H_{r-1} \neq 0$ **then**
  $\rho_r :=$`ExtendRepresentation`$(\rho_{r-1}, L)$;
  $\sigma :=$ `DirectSum`$(\rho_r, \mathrm{ad})$;
**else**
  $\sigma := \rho_{r-1}$;
**fi**;

**Proof.** We start with a representation of the commutative subalgebra $K_1$. We remark that $\rho_1(x)$ is nilpotent for all $x \in \mathrm{NR}(K_1) = K_1$. Then we successively construct representations $\rho_i$ of $K_i$. By Lemma 4.2 an invariant of the process is that $\rho_i(x)v_1 = 0$ for $x \in K_i$. Hence we have the correct input for the subroutine `ExtendRepresentation`. Also by Proposition 3.2 and property 3 of the output of `ExtensionSeries` we have that $\rho_i$ is always nilpotent on the nilradical of $K_i$ so that $S_{\rho_i}$

will be finite dimensional. By Proposition 4.1 and property 4 of the output of `ExtensionSeries` we have that `ExtendRepresentation` will each time return a faithful representation.

The last step is the extension to the Lie algebra $R \rtimes S$, where $R$ is the solvable radical of $L$ and $S$ is a Levi subalgebra. This time after having called the `ExtendRepresentation` there is no guarantee that the resulting representation will be faithful. However it will be faithful on $R$ and consequently on the centre of $L$. Then we take the direct sum with the adjoint representation obtaining a representation that is faithful on the centre as well as on the rest of $L$. $\square$

**Corollary 5.1 (Ado's theorem)** *Let $L$ be a finite dimensional Lie algebra over a field of characteristic zero. Then $L$ has a faithful finite dimensional representation. Moreover, a representation can be constructed such that the elements of the nilradical of $L$ are represented by nilpotent matrices.*

**Proof.** Set $\sigma = \texttt{Representation}(L)$. Then by Proposition 3.2 we have that $\sigma$ is nilpotent on $\mathrm{NR}(L)$. $\square$

Now we consider bounding the degree (i.e., the dimension of the representation space) of the representation produced. In general we are not able to do this, however if $L$ is nilpotent then we can prove a bound. For this we introduce a weight function $w$ on $U(L)$, following [2]. So let $L$ be a nilpotent Lie algebra of nilpotency class $c$. This means that the lower central series of $L$ is

$$L = L^1 \supset L^2 \supset \cdots \supset L^c \supset L^{c+1} = 0.$$

Then for $x \in L$ we let $w(x)$ be the number $k$ such that $x \in L^k$ but $x \notin L^{k+1}$. We extend $w$ to $U(L)$ by setting $w(ab) = w(a) + w(b)$ and $w(a+b) = \min(w(a), w(b))$ if $a + b \neq 0$. Furthermore we set $w(1) = 0$ and $w(0) = \infty$. Let $K_1 \subset K_2 \subset \cdots \subset K_r = L$ be the series constructed in the algorithm `ExtensionSeries`. Then $K_{i+1} = K_i \rtimes \langle y_i \rangle$ and $K_1 = \langle x_1, \ldots, x_s \rangle$. Let $\rho_1$ be a representation of $K_1$ given by $\rho_1(x_i) = e^{s+1}_{1,i+1}$. Then by successively extending $\rho_1$ we obtain representations $\rho_i$ of $K_i$.

**Proposition 5.2** *Suppose that $\rho_i(a) = 0$ for every element $a \in U(K_i)$ such that $w(a) \geq c+1$. Then $\rho_{i+1}(b) = 0$ for all $b \in U(K_{i+1})$ such that $w(b) \geq c+1$. Furthermore $f(b) = 0$ for all $f \in S_{\rho_i}$ and $b \in U(K_i)$ such that $w(b) \geq c+1$.*

**Proof.** For the first statement let $b \in U(K_{i+1})$ be a monomial such that $w(b) \geq c+1$. According to the construction of $\rho_{i+1}$ there are two cases to be considered.

First we consider the case where there is a $\tilde{y}_i \in K_i$ such that $y_i - \tilde{y}_i \in Z(K_{i+1})$. This means that $\rho_{i+1}$ is constructed in the first part of `ExtendRepresentation`. After replacing $y_i$ by $y_i - \tilde{y}_i$ we may suppose that $\tilde{y}_i = 0$. Then $\rho_{i+1}(y_i)\rho_{i+1}(a) = \rho_{i+1}(a)\rho_{i+1}(y_i) = 0$ for all $a \in U(K_i) \setminus \{1\}$. Now if $b$ contains a $y_i$, then $\rho_{i+1}(b) = 0$. Otherwise $b$ is also an element of $U(K_i)$ and again from the construction of $\rho_{i+1}$ it is seen that $\rho_{i+1}(b) = 0$.

Now we consider the case where $\rho_{i+1}$ is constructed by `GeneralExtension`. Let $a$ be an element of $U(K_i)$, then we claim that $w(y_i a - a y_i) \geq w(a) + w(y_i)$. First we have $w(a y_i) = w(y_i a) = w(a) + w(y_i)$. So if $y_i a - a y_i \neq 0$, then the claim follows from the fact that the weight of a sum is the least of the weights of its terms. On the other hand, if $y_i a - a y_i = 0$ then its weight if $\infty$. Let $f = y_i^l \cdot \theta(v_p, v_r^*)$ be an element of $S_{\rho_i}$, where $l \geq 0$. Then for $b' \in U(K_i)$ we calculate

$$(b \cdot f)(b') = \pm v_r^*(\rho_i(y_i^l \cdot (b \cdot b'))v_p).$$

Here $y_i \cdot g = y_i g - g y_i$ and $x \cdot g = gx$ for $x \in K_i$ and $g \in U(K_i)$. Now from our claim above it follows that $w(y_i^l \cdot (b \cdot b')) \geq w(b)$. (Note that $y_i^l \cdot (b \cdot b')$ lies in $U(K_i)$ whereas $b$ lies in $U(K_{i+1})$.) Hence $\rho_i(y_i^l \cdot (b \cdot b')) = 0$ so that $b \cdot f = 0$. Now by Lemma 4.3 we have that $\rho_{i+1}(b) = 0$.

For the second statement let $b \in U(K_i)$ be an element such that $w(b) \geq c+1$ and let $f = y_i^l \cdot \theta(v_p, v_r^*)$ be an element of $S_{\rho_i}$. Then $f(b) = \pm v_r^*(\rho_i(y_i^l \cdot b)v_p)$, which is 0 because $w(y_i^l \cdot b) \geq w(b) \geq c+1$. $\square$

**Corollary 5.3** *Let $L$ be a nilpotent Lie algebra of dimension $n$ and nilpotency class $c$. Set*

$$\sigma = \texttt{Representation}(L).$$

*Then the degree of $\sigma$ is bounded above by $\binom{n+c}{c}$.*

**Proof.** Since $\sigma = \rho_r$ we have that the representation space of $\sigma$ is $S_{\rho_{r-1}}$. The representation $\rho_1$ of $K_1$ satisfies the requirement of Proposition 5.2. The conclusion is that $f(b) = 0$ for $f \in S_{\rho_i}$, $b \in U(K_i)$ such that $w(b) \geq c+1$ and $i = 1, \ldots, r-1$. It follows that the degree of $\sigma$ is bounded above by the number of monomials in $U(L)$ of degree at most $c$. Now the number of monomials of degree $d$ in $U(L)$ is given by $\binom{n+d-1}{d}$. We introduce an auxiliary variable $z$ and if $a \in U(L)$ is a monomial of degree $d \leq c$, then we associate to it the expression $az^{c-d}$. It is seen that the number of monomials of $U(L)$ of degree $\leq c$ is bounded above by the number of monomials in $n+1$ variables of degree exactly $c$. Hence the statement follows. $\Box$

**Remark.** Since the maximal nilpotency class of a Lie algebra of dimension $n$ is $c = n-1$, we have that the bound of Corollary 5.3 in general is exponential in $n$. However, for Lie algebras of constant nilpotency class, the bound is polynomial in $n$.

**Remark.** If the field over which $L$ is defined is of characteristic $p > 0$, then $L$ might not have a Levi decomposition. However, if $L$ has a Levi decomposition, then the algorithm will yield a representation for $L$ also in this case.

## 6 Examples, and practical experiences

**Example 6.1** Let $L = K \rtimes \langle y \rangle$, where $K$ is a commutative subalgebra spanned by $\{x_1, \ldots, x_t\}$. We suppose that $L$ is not commutative and try to find a representation of $L$. We start with a representation $\rho$ of $K$ given by $\rho(x_i) = e^{t+1}_{1,i+1}$. Then one extension step will yield a representation of $L$. First we calculate $C_\rho$. For $i > 1$ we have

$$\theta(v_i, v_j^*)(a) = v_j^*(\rho(a)v_i) = \delta_{j,1} f_{x_{i-1}}(a) + \delta_{i,j} f_1(a).$$

So a basis of $C_\rho$ is given by

$$\{f_1, f_{x_1}, \ldots, f_{x_t}\}.$$

We suppose that $[y, x_i] = \sum_j c_{ij} x_j$ and we calculate the action of $y$ on $C_\rho$:

$$y \cdot f_{x_i}(a) = -f_{x_i}(ya - ay) = \sum_{k=1}^{t} -c_{ki} f_{x_k}(a).$$

| $n$ | $\dim L_n$ | nilpotency class | Degree | Runtime (s) |
|---|---|---|---|---|
| 3 | 3 | 2 | 3 | 2 |
| 4 | 6 | 3 | 7 | 34 |
| 5 | 10 | 4 | 16 | 710 |
| 6 | 15 | 5 | 35 | 7410 |

Table 1: Degrees of the representation of the Lie algebra $L_n$ found by the algorithm `Representation`. The last column displays the runtime of the process in seconds.

It follows that $C_\rho$ is already a module for $L$. The values of the representation $\sigma : L \longrightarrow \mathfrak{gl}(C_\rho)$ are given by $\sigma(x_i) = e^{t+1}_{1,i+1}$ and $\sigma(y) = -(\mathrm{ad}\, y)^T$.

**Example 6.2** Let $L = \mathfrak{gl}_n(F)$, the Lie algebra of all $n \times n$ matrices. The Levi decomposition of this Lie algebra is $L = \langle x \rangle \rtimes K$, where $K \cong \mathfrak{sl}_n(F)$ and $\langle x \rangle = Z(L)$. Then we start with a representation $\rho$ of the 1-dimensional Lie algebra $\langle x \rangle$, given by $\rho(x) = e^2_{1,2}$. Now $C_\rho = \{f_1, f_x\}$. Since $K$ commutes with $\langle x \rangle$ we have that $C_\rho$ is a trivial module for $L$. Hence in this case we need to take the direct sum with the adjoint representation of $L$, obtaining a representation of degree $n^2 + 2$.

We implemented the algorithm using a library of routines that operate on finite dimensional Lie algebras called ELIAS (for Eindhoven LIe Algebra System). This library will be a part of GAP4. We tried [1] the method on the Lie algebras $L_n$ of strictly upper triangular matrices of order $n$, for $n = 3, 4, 5, 6$. The degrees of the resulting representations are shown in Table 1. It is seen that the resulting degree is much less than the bound provided by Corollary 5.3. However the algorithm seems to have an exponential behaviour. So for nilpotent Lie algebras of small dimension the algorithm works fine. But when the dimension and the nilpotency class increase, the algorithm might become slow.

## References


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