Riemannian foliations of symmetric spaces

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- **Meta-question 1** classify the Riemannian foliations whose leaves satisfy a "natural" geometric property.

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- Explicit classification if $rank(M) \leq 2$.

Hypersurfaces of Symmetric spaces

Theorem

(Levi-Civita, Somigiliana, Segre) A hypersurface $M \subset \mathbb{R}^n$ has constant principal curvatures if and only if it is locally isometric to either a hypersphere, a cylinder,or a plane.

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- Inhomogeneous examples?

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- Cartan: M is isoparametric $\Leftrightarrow$ all parallel hypersurfaces have constant mean curvatures.

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- For $g = 4$ inhomogeneous examples of FKM type arising from Clifford systems.
- Wang: Under the Hopf fibration inhomogeneous isoparametric hypersurfaces project to isoparametric hypersurfaces with nonconstant principal curvatures.

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  2. Complex submanifolds of $\mathbb{C}P^n$ and $\mathbb{C}H^n$.
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- Every tube around a curvature-adapted submanifold is curvature-adapted.

Theorem

(M.) Let M be a curvature-adapted hypersurface of a compact symmetric space. Then M is isoparametric if, and only if, M has constant principal curvatures and the eigenvalues of $K_\xi(E)$ are constant.

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Theorem

(M.) Let $M \subset \mathbb{O}P^2$ be a complete curvature-adapted hypersurface. Then it is isoparametric if, and only if it is homogeneous.

- Fundamental techniques are given by using the Riccati equation for the tube around $M \subset \overline{M}$:

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- Focal point: principal curvatures developing singularities.

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- For $M_d \subset \mathbb{C}P^n$,
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- Motivated by this work, Gray established the formula

$$minfoc_{\mathbb{C}P^n}(M_d^{n-1}) = r_0 \leq ArcSin(\frac{1}{\sqrt{d}}).$$

Theorem

(M.) A complex submanifold M arises as an exceptional leaf of any Riemannian foliation $\mathcal{F}^{2n-1}$ on $\mathbb{C}P^n$ if, and only if, M is isometric to

1. *a totally geodesic $\mathbb{P}^k \subset \mathbb{P}^n$ for some $k \in \{0, \ldots, n-1\}$,*
2. *the complex quadric $Q^{n-1} = \{[z] \in \mathbb{P}^n : z_0^2 + \cdots + z_n^2 = 0\} \subset \mathbb{P}^n$,*
3. *the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^k$ into $\mathbb{P}^{2k+1}$,*
4. *the Plücker embedding of the complex Grassmann manifold $G_2(\mathbb{C}^5)$ into $\mathbb{P}^9$, or*
5. *the half spin embedding of $SO(10)/U(5)$ in $\mathbb{P}^{15}$.*

In other words, $\mathcal{F}$ is induced by a cohomogeneity one action.

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- Others?

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- Idea of the proof: use infinitesimal Riemannian foliation $T_x\mathcal{F}$.
- At a singular point $T_x\mathcal{F} = T_x\mathcal{F}^{triv} \oplus T_x\mathcal{F}^{ess}$
- $T_x\mathcal{F}^{ess}$ is an isoparametric foliation of Euclidean space $\longrightarrow$ homogeneous.

Theorem

(M.-Nagy) Let $\mathcal{F}$ be a Riemannian foliation of an open subset $\mathcal{O} \subset \mathbb{C}P^n$. If it is complex, it is $\mathcal{F}_{Tw}|_{\mathcal{O}}$.

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Theorem

(M.-Nagy) Let $\mathcal{F}$ be a complex Riemannian foliation of a closed irreducible Hermitian symmetric space $\overline{M}$. Then $\mathrm{rank}(M) \leq 3$, and the leaves of $\mathcal{F}$ are totally geodesic $\mathbb{C}P^1 \subset \overline{M}$.

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