Maximal abelian Borel stable subspaces and the CDSW conjecture for symmetric spaces

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The main problem

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σ indecomposable involution of $\mathfrak{g}$
$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ eigenspace decomposition

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graded case: $\mathfrak{g}$ simple.

Problem

Fix a Borel subalgebra $\mathfrak{b}_0 \subset \mathfrak{g}_0$. Find the maximal dimension of a $\mathfrak{b}_0$-stable abelian subalgebra of $\mathfrak{g}_1$.

It turned out that it was more natural to try solving the following more general problem.

Problem'

Find the poset structure of certain special posets $\mathcal{I}_{\alpha,\mu}$ of $\mathfrak{b}_0$-stable abelian subalgebra of $\mathfrak{g}_1$.

The CDSW conjecture

Let $\mathfrak{g}$ be simple. Consider the $\mathfrak{g}$ – *map* $c : \mathfrak{g} \to \bigwedge^2 \mathfrak{g}$, $c(x) = \sum_{i=1}^{\dim \mathfrak{g}} [x, x_i] \wedge x^i$, where $\{x_i\}$, $\{x^i\}$ are a pair of dual bases w.r.t. the Killing form of $\mathfrak{g}$. Set $R = \bigwedge(\mathfrak{g} \oplus \mathfrak{g})$. Using c, can define three $\mathfrak{g}$-maps with target R,

$$c_1 : \mathfrak{g} \to \bigwedge^2 \mathfrak{g} \otimes 1,\ c_2 : \mathfrak{g} \to 1 \otimes \bigwedge^2 \mathfrak{g},\ c_3 : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}.$$

Define

$$B = R/\langle Im(c_1), Im(c_2)\rangle, \quad A = R/\langle Im(c_1), Im(c_2).Im(c_3)\rangle.$$

Cachazo-Douglas-Seiberg-Witten conjecture

If S is the image in A of $\sum_i x_i \otimes x^i$, then $A^{\mathfrak{g}}$ is a truncated polynomial algebra of dimension $h^\vee$, $h^\vee$ being the dual Coxeter number of $\mathfrak{g}$.

The CDSW conjecture for symmetric spaces

Kumar has version of the conjecture for infinitesimal symmetric spaces $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Just mimic the above construction using the $\mathfrak{g}_0 - map\ \tilde{c} : \mathfrak{g}_0 \to \bigwedge^2 \mathfrak{g}_1$, $\tilde{c}(x) = \sum_{i=1}^{\dim \mathfrak{g}} [x, x_i] \wedge x^i$, where $\{x_i\}$, $\{x^i\}$ are a pair of dual bases of $\mathfrak{g}_1$.Correspondingly, set $\tilde{A} = \bigwedge(\mathfrak{g}_1 \oplus \mathfrak{g}_1)/\langle Im(\tilde{c}_1), Im(\tilde{c}_2).Im(\tilde{c}_3)\rangle$ and $\tilde{S} = \sum_{i=1}^{\dim \mathfrak{g}_1} x_i \otimes x^i$

Proposition

If $\mathfrak{g}_1$ is irreducible as a $\mathfrak{g}_0$-module, then $\tilde{A}^{\mathfrak{g}}$ is a truncated polynomial algebra generated by $\tilde{S}$.

Speculation

Recall the special posets $\mathcal{I}_{\alpha,\mu}$ which appeared (undefined!) a few slides ago. We prove that they have minimum.

Speculation

$$\dim \tilde{A}^{\mathfrak{g}} = \text{nilpotency class of } \tilde{S} = \min_{\alpha,\mu}(\dim \min \mathcal{I}_{\alpha,\mu}).$$

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The above statement turns out to be true

- in the adjoint case, where it reduces to the CDSW conjecture;
- in the graded case with $\mathfrak{g}_0, \mathfrak{g}_1$ simple, where it reduces to a conjecture of Kumar;
- for any g, σ with g_1 irreducible such that $\dim \mathfrak{g}_1 \leq 16$ (MAGMA computations, done with the help of John Cannon).

Problem': a short historical perspective

- *Schur 1905*: There exist at most $\lfloor \frac{N^2}{4} \rfloor + 1$ linearly independent commuting matrices in $gl(N)$.

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Generalization to the graded case

Abelian Case

- Kostant's Theorem
- Peterson's Theorem

- Panyushev's bijection between maximal abelian ideals and long simple roots
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$\mathbb{Z}_2$-graded case

- Panyushev
- Cellini-Möseneder-P.

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Notation

If $\mathfrak{a} = \oplus_{i=1}^{k} \mathbb{C} v_i$ is an abelian subalgebra of $\mathfrak{g}$, set

$$v_{\mathfrak{a}} = v_1 \wedge \ldots \wedge v_k \in \bigwedge^k \mathfrak{g}.$$

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- $\mathcal{C}$ Casimir element (w.r.t. the Killing form $\mathfrak{g}$).
- m_k is the maximal eigenvalue of $\mathcal{C}$ on $\bigwedge^k \mathfrak{g}$
- M_k eigenspace of $\mathcal{C}$ on $\bigwedge^k \mathfrak{g}$ of eigenvalue k
- $A_k = Span(v_{\mathfrak{a}} \mid \mathfrak{a}$ abelian, $\dim(\mathfrak{a}) = k)$

Kostant Theorems

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$A_k = Span(v_\mathfrak{a} \mid \mathfrak{a}$ abelian, $\dim(\mathfrak{a}) = k)$

Theorem (Kostant 1965)

1. *$m_k \leq k$, and $m_k = k$ iff $A_k \neq \emptyset$*
2. *$A := \sum_k A_k$ is a multiplicity-free $\mathfrak{g}$-module. Its highest weights are parametrized by the set $\mathcal{A}b$ of abelian ideals of a Borel subalgebra.*
3. *$\bigwedge(\mathfrak{g}) = A \oplus \langle Im(c) \rangle$.*

Peterson Enumeration Theorem

$\mathfrak{i}$ abelian ideal of $\mathfrak{b}$.

$$\mathfrak{i} = \bigoplus_{\alpha \in \Phi_{\mathfrak{i}}} \mathfrak{g}_\alpha$$

$\Phi_{\mathfrak{i}} \subset \Delta^+$ "abelian "dual order ideal of the root poset.

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$$\langle \mathfrak{i} \rangle = \sum_{\alpha \in \Phi_{\mathfrak{i}}} \alpha.$$

Then

$$A \cong \bigoplus_{\mathfrak{i}} L(\langle \mathfrak{i} \rangle)$$

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The cardinality of the set of abelian ideals of $\mathfrak{b}$ is $2^{\mathrm{rank}(\mathfrak{g})}$.

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Main idea

Bijection

$$\Phi_{\mathfrak{i}} \subset \Delta^+ \longleftrightarrow w_{\mathfrak{i}} \in \widehat{W} \text{ s.t. } N(w_{\mathfrak{i}}) = \{\delta - \alpha \mid \alpha \in \Phi_{\mathfrak{i}}\}$$

where $N(w) = \{\alpha \in \widehat{\Delta}^+ \mid w^{-1}(\alpha) \in -\widehat{\Delta}^+\}$.

Proof of Peterson Theorem (Cellini-P.).

C_1 fundamental alcove,
$\mathcal{W}^{ab} = \left\{ w \in \widehat{W} \mid N(w) = \{\delta - \alpha \mid \alpha \in \mathfrak{i}\}, \mathfrak{i} \in \mathcal{A}b \right\}$.
We prove that

$$w \in \mathcal{W}^{ab} \iff wC_1 \subset 2C_1$$

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In particular

$$2C_1 = \bigcup_{w \in \mathcal{W}^{ab}} wC_1$$

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Fact *If* $w \in \mathcal{W}^{ab}$, *then* $w^{-1}(-\theta + 2\delta) \in \Delta_\ell^+$.

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$$0 < w^{-1}(-\theta + 2\delta) = w^{-1}(-\theta + \delta) + \delta$$

$$= -k\delta + \gamma + \delta = \begin{cases} k > 1 & \text{impossible} \\ k = 0 & \text{excluded} \\ k = 1 & \implies \gamma \in \Delta_\ell^+ \end{cases}$$

Panyushev bijection

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\end{aligned}
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So one defines

$$\mathcal{I}_\alpha = \{\mathfrak{i} \in \mathcal{A}b \mid w_{\mathfrak{i}}^{-1}(-\theta + 2\delta) = \alpha\}.$$

Clearly

$$\mathcal{A}b' = \coprod_{\alpha \in \Delta_\ell^+} \mathcal{I}_\alpha$$

Panyushev bijection

Proposition (Panyushev, Suter)

$$\mathcal{I}_\alpha \cong \widehat{W}_\alpha / W_\alpha$$

where

$$\widehat{W}_\alpha = \langle s_\beta \mid \beta \in \widehat{\Pi}, \beta \perp \alpha \rangle$$
$$W_\alpha = \langle s_\beta \mid \beta \in \Pi, \beta \perp \alpha \rangle$$

In particular, $\mathcal{I}_\alpha$ has minimum and maximum. Moreover, the map

$$\beta \mapsto \max \mathcal{I}_\beta$$

sets up a bijection among long simple roots and maximal abelian ideals.

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The graded case: setting

- $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$
- $\mathfrak{h}_0 \subset \mathfrak{g}_0$ Cartan, Δ_0 roots, Δ_0^+ positive r., Π_0 simple r.
- $\widehat{L}(\mathfrak{g}, \sigma) = \sum_{i \in 2\mathbb{Z}} t^j \otimes \mathfrak{g}_0 \oplus \sum_{i \in 2\mathbb{Z}+1} t^j \otimes \mathfrak{g}_1 \oplus \mathbb{C}c \oplus \mathbb{C}d$
- $\widehat{\mathfrak{h}} = \mathfrak{h}_0 \oplus \mathbb{C}c \oplus \mathbb{C}d$ Cartan, $\widehat{\Delta}$ roots
- $\widehat{\Delta}^+ = \Delta^+ \cup \{\alpha \in \widehat{\Delta} \mid \alpha(d) > 0\}$ positive roots
- $\widehat{\Pi} = \{\alpha_0, \ldots \alpha_n\}, n = \mathit{rank}\, \mathfrak{g}_0$, simple roots
- $\widehat{W}$ Weyl group of $\widehat{L}(\mathfrak{g}, \sigma)$.

Kac classification of finite order automorphisms

Let $\mathfrak{g}$ be a simple Lie algebra of type X_N.

Theorem (Kac)

Up to conjugation in $Aut(\mathfrak{g})$, an automorphism σ of order m can be encoded by an $(n+2)$-ple

$$(s_0, \dots, s_n, k) : m = k \sum_{i=0}^{n} m_i s_i,$$

where $n = rank(\mathfrak{g}^\sigma)$, k is the minimal integer s.t. σ^k is inner, $s_0, \dots, s_n$ are positive coprime integers and $m_0, \dots, m_n$ are the labels of the Dynkin diagram of type $X_N^{(k)}$.

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When $m = 2$

1. $k = 1$, $\exists p$ s.t. $s_p = 2$ and $s_i = 0$ if $i \neq p$;
   $k = 2$, $\exists p$ s.t. $s_p = 1$ and $s_i = 0$ if $i \neq p$;
2. $k = 1$, $\exists p, q$ s.t. $s_p = s_q = 1$ and $s_i = 0$ if $i \neq p \neq q$.

Encoding the b_0-stable abelian subspaces: notation

If $\gamma = \sum_i c_i \alpha_i \in \widehat{\Delta}^+$ then we can define $ht_\sigma(\gamma) = \sum_i s_i c_i$.

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We say that $w \in \widehat{W}$ is *σ-minuscule* if

$$N(w) \subset \{\gamma \in \widehat{\Delta}^+ \mid ht_\sigma(\gamma) = 1\}.$$

The set of σ-minuscule elements is denoted by $\mathcal{W}_\sigma^{ab}$.

Encoding the b_0-stable abelian subspaces: theorems

Theorem (Cellini-Möseneder-P.)

If $w \in \mathcal{W}_\sigma^{ab}$, $N(w) = \{\beta_1, \dots, \beta_k\}$, then the map $Ab : \mathcal{W}_\sigma^{ab} \to \mathcal{I}_{ab}^\sigma$ given by

$$Ab(w) = \bigoplus_{i=1}^{k} (\mathfrak{g}_1)_{-\bar{\beta}_i}$$

(where $\lambda \mapsto \bar{\lambda}$ is the restriction map $\widehat{\mathfrak{h}} \to \mathfrak{h}_0$), is a poset isomorphism.

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The proof is based on Garland-Lepowsky generalization to the affine case of Kostant's theorem on the homology of the nilpotent radical.

Encoding the b_0-stable abelian subspaces: theorems

Theorem (Cellini-Möseneder-P.)

Set

$$D_\sigma = \bigcup_{w \in \mathcal{W}_\sigma^{ab}} wC_1.$$

D_σ *is a polytope. More precisely,* $D_\sigma = \cap_{\alpha \in \Phi_\sigma} H_\alpha^+$ *where*

$$\Phi_\sigma = \begin{cases} \widehat{\Pi}_0 \cup \{\alpha_i + ks_i\delta \mid \alpha_i \in \widehat{\Pi}\} & \text{in "most" cases,} \\ \widehat{\Pi}_0 & \text{in the other cases.} \end{cases}$$

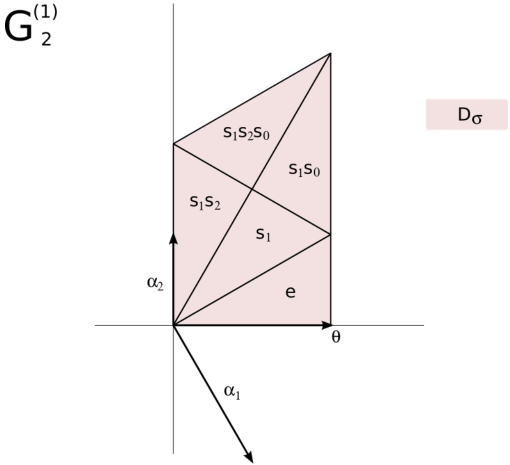
Here

$$\widehat{\Pi}_0 = \Pi_0 \cup \cup_\Sigma \{k\delta - \theta_\Sigma\}$$

Example of D_σ

Let σ be the inner involution of G_2 with $\mathfrak{g}_0 \cong A_1 \times A_1$. Equations for D_σ are

$$\alpha_1 \geq -1, \quad 0 \leq \alpha_2 \leq 1, \quad 0 \leq \theta \leq 1.$$



Looking for maximal elements in $\mathcal{W}_\sigma^{ab}$

Set

$$\mathcal{M}_\sigma = \Phi_\sigma \setminus (\widehat{\Pi} \cap \Phi_\sigma)$$

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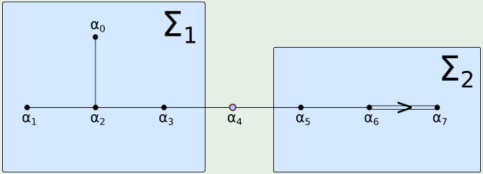
Example ($B_7^{(1)}$, $\mathfrak{g}_0 \cong D_4 \times B_3$)



$$\begin{aligned}
\Pi_0 &= \Sigma_1 \cup \Sigma_2, \quad \Pi_1 = \{\alpha_4\} \\
\mathcal{M}_\sigma &= \{k\delta - \theta_{\Sigma_1}, k\delta - \theta_{\Sigma_2}\} \cup \{k\delta + \alpha_4\} \\
&= \{\delta - \alpha_1 - \alpha_0 - 2\alpha_2 - \alpha_3, \delta - \alpha_5 - 2\alpha_6 - 2\alpha_7, \delta + \alpha_4\}
\end{aligned}$$

Looking for maximal elements in $\mathcal{W}_\sigma^{ab}$

Proposition

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So we are led to introduce the following posets

$$\mathcal{I}_{\alpha,\mu} = \{w \in \mathcal{W}_\sigma^{ab} \mid w(\alpha) = \mu\}$$

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1. When $\mathcal{I}_{\alpha,\mu}$ is nonempty ?

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Answer via a combinatorial criterion:

$$\mathcal{I}_{\alpha,\delta-\theta_\Sigma} \neq \emptyset \iff \alpha \in A(\Sigma)_\ell$$

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$$\mathcal{I}_{\alpha,\delta-\theta_\Sigma} \cong \widehat{W}_\alpha / \widehat{W}'_\alpha$$

where $\widehat{W}'_\alpha$ is a reflection subgroup of $\widehat{W}_\alpha$ whose structure depends on σ

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It turns out that the intersection is non void exactly when $\alpha \in \Sigma'$, $\beta \in \Sigma$

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Recall that $\mathcal{I}_\alpha \cong \widehat{W}_\alpha / \widehat{W}'_\alpha$. We show that when $\widehat{W}'_\alpha$ is not standard parabolic, maximal elements appear in pairs of $\mathcal{I}_{\alpha,\mu}$'s.

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5. When maximal elements of $\mathcal{I}_{\alpha,\mu}$ are global maxima ?

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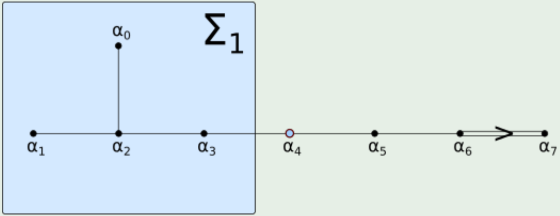
5. When maximal elements of $\mathcal{I}_{\alpha,\mu}$ are global maxima ?
   Answer: almost always!

The structure of $\mathcal{I}_{\alpha,\mu}$: Non-emptiness

Example ($B_7^{(1)}$)



$$\mu = \delta - \theta_{\Sigma_1}, \quad \alpha \in A(\Sigma_1)_\ell$$

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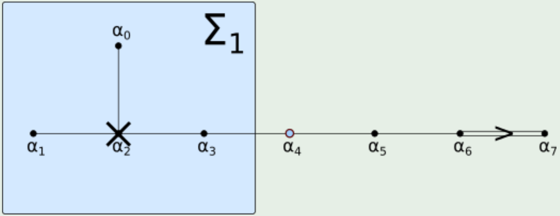

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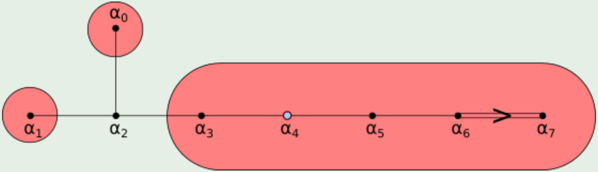

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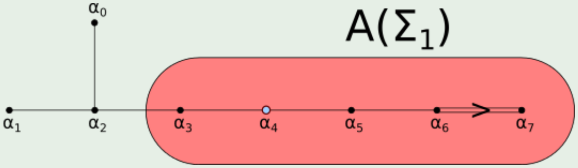

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The structure of $I_{\alpha,\mu}$: Non-emptiness

Proof of non-emptiness.

We show that $I_{\alpha,\mu}$ has a minimum $w_{\alpha,\mu}$. $\square$

Theorem (Cellini-Möseneder-P.-Pasquali 2011)

The maximal $\mathfrak{b}_0$-stable abelian subalgebras are parametrized by the set

$$\mathcal{M} = \left(\bigcup_{\substack{\Sigma \mid \Pi_0 \\ \Sigma \text{ of type 1}}} (A(\Sigma) \cap \Sigma)_\ell \right) \cup \left(\bigcup_{\substack{\Sigma \mid \Pi_0 \\ \Sigma \text{ of type 2}}} \Sigma_\ell \right) \cup$$
$$\cup \left(\bigcup_{\substack{\Sigma, \Sigma' \mid \Pi_0,\, \Sigma \prec \Sigma' \\ \Sigma, \Sigma' \text{ of type 1}}} (\Sigma_\ell \times \Sigma'_\ell) \right) \cup \Pi_1^1.$$

Example

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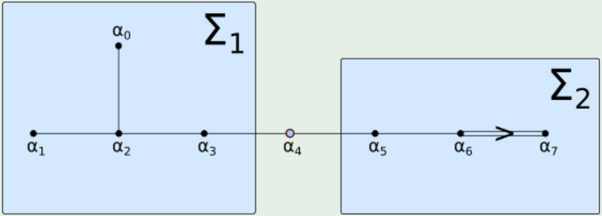

$$\begin{aligned}\mathcal{M} \quad = \quad & \{\alpha_3, \alpha_5, (\alpha_1, \alpha_5), (\alpha_1, \alpha_6), (\alpha_2, \alpha_5), (\alpha_2, \alpha_6), (\alpha_3, \alpha_5), \\ & (\alpha_3, \alpha_6), (\alpha_0, \alpha_5), (\alpha_0, \alpha_6), \alpha_4\}\end{aligned}$$

Back to the CDSW conjecture

- $\mathbf{G}$ connected simply connected group corresponding to $\mathfrak{g}$
- $\mathbf{B}$ Borel subgroup corresponding to $\mathfrak{b}$
- $\mathcal{G}$ Kac-Moody group corresponding to $\widehat{\mathfrak{g}}$, i.e. central extension of the loop group $L(\mathbf{G}) = \mathbf{G}(\mathbb{C}[[\mathbf{t}]][\mathbf{t}^{-1}])$
- $\mathcal{P}$ parabolic corresponding to $\mathfrak{g} \subset \widehat{\mathfrak{g}}$, i.e. central extension of $\mathbf{G}(\mathbb{C}[[\mathbf{t}]])$
- $\mathcal{Y} = \mathcal{G}/\mathcal{P}$ affine Grassmannian, $\mathcal{B} \subset \mathcal{P}$ standard Borel.

Back to the CDSW conjecture

Bruhat decomposition:

$$\mathcal{Y} = \coprod_{w \in W'} \mathcal{B} w \mathcal{P}/\mathcal{P}$$

where W' are the dominant elements in $\widehat{W}$, i.e., wC_1 is in the fundamental chamber.

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Define

$$\mathcal{Y}_2 = \coprod_{w \in \mathcal{W}^{ab}} \mathcal{B} w \mathcal{P}/\mathcal{P},$$

a closed subvariety of $\mathcal{Y}$.

Back to the CDSW conjecture

Recall that

$$B^{\mathfrak{g}} = \left(\bigwedge \mathfrak{g}/\langle Im(c)\rangle \otimes \bigwedge \mathfrak{g}/\langle Im(c)\rangle\right)^{\mathfrak{g}}$$

has a basis $\{z_w\}$ indexed by $w \in \mathcal{W}^{ab}$: if $w = w_{\mathfrak{i}}$, $\{x_i\}$ is a basis of the submodule of $\bigwedge \mathfrak{g}$ generated by $v_{\mathfrak{i}}$ and $\{y_i\}$ is the dual basis, set

$$z_w = \bigwedge_i x_i \wedge y_i.$$

Theorem (Kumar)

The map $H^*(\mathcal{Y}_2) \to B^{\mathfrak{g}}$ mapping the Schubert basis ε_w to z_w is a graded algebra isomorphism.

Back to the CDSW conjecture

$\mathcal{Y}_2 \hookrightarrow \mathcal{Y}$ gives a restriction map $H^*(\mathcal{Y}) \to H^*(\mathcal{Y}_2)$. Recall the projection $\pi : B^{\mathfrak{g}} \to A^{\mathfrak{g}}$. We have

$$\mathbb{C}[p_1, \dots, p_n] = S(\mathfrak{g}^*)^{\mathfrak{g}} = H^*(\Omega_e(\mathbf{K})) = H^*(\mathcal{Y}) \to H^*(\mathcal{Y}_2) = B^{\mathfrak{g}} \to A^{\mathfrak{g}}$$

where $2 = \deg p_1 \leq \dots \leq \deg p_n$.

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Theorem (Kumar)

$$p_1 \mapsto S, \qquad p_i \mapsto Ker\pi,\ i > 1.$$

In particular, $A^{\mathfrak{g}}$ is generated by S

Comment on the "speculation"

Proposition

In the graded setting, in $B^{\mathfrak{g}}$ we have

$$S^k = \pm \sum_{w \in \mathcal{W}_\sigma^{ab}, \ell(w)=k} z_w.$$

If $g = \min_{\alpha,\mu} \ell(w_{\alpha,\mu})$, then at least one summand z_w in S^g is such that wC_1 has a face on a wall in $\mathcal{M}_\sigma$ and g is minimal with this property. Therefore S^{g+1}, viewed in $H^*(\mathcal{Y})$, has some component outside D_σ, so projecting down to $H^*(\mathcal{Y}_2)$ this component goes to 0. **This should imply that S^{g+1} goes to $Ker\pi$; this would prove the full conjecture in the graded case**.