Hamiltonian characteristic classes and the cohomology of the group of hamiltonian symplectomorphisms of some homogeneous spaces

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Symplectic manifolds

$$(M^{2n}, \omega),\ d\omega = 0$$

ω is a non-degenerate 2-form.

Example 1: $\mathbb{R}^{2n}$, with coordinates $\{x_1, y_1, \cdots x_n, y_n\}$

$$\omega = dx_1 \wedge dy_1 + \cdots dx_n \wedge dy_n.$$

Example 2: $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$, with the same formula for ω

Example 3: orientable surfaces: $S^2, \mathbb{T}^2, \Sigma_g,\ g > 1.$

Coadjoint orbits

- Lie group G, its Lie algebra $\mathfrak{g}$, the dual $\mathfrak{g}^*$
- adjoint representation: $Ad\, g : \mathfrak{g} \to \mathfrak{g}$,
- coadjoint representation $(Ad\, g)^* : \mathfrak{g}^* \to \mathfrak{g}^*$
- on the Lie algebra level:

$$ad\, X : \mathfrak{g} \to \mathfrak{g},\ ad\, X(Y) := [X, Y],$$

$$(ad\, X)^* : \mathfrak{g}^* \to \mathfrak{g}^*.$$

Coadjoint orbit:

$$\mathcal{O}(\eta) = \{(Ad\, g^{-1})^*(\eta) \mid g \in G\}.$$

Coadjoint orbits are symplectic

- $\mathcal{O}(\eta) \cong G/G_\eta$,
- $T_\eta\mathcal{O}(\eta) = \{(ad\, X)^*\eta, \mid X \in \mathfrak{g}\}$
- $\omega_\eta((ad\, X)^*\eta, (ad\, Y)^*\eta) := \langle \eta, [X, Y] \rangle$

Example - S^2

S^2 is diffeomorphic to the coadjoint orbit in $so(3)^*$

Complex Grassmannian $U(m+n)/U(m) \times U(n)$

The adjoint orbit of $diag(i, \cdots i, -i, \cdots - i)$ in $su(m+n)$ is $U(m+n)/U(m) \times U(n)$.

In particular, complex projective space $\mathbb{C}P^n$

$\mathbb{C}P^n = U(n+1)/U(1) \times U(n)$ is a coadjoint orbit.

Symplectomorphism group $\mathrm{Symp}(M, \omega)$

$$\mathrm{Symp}(M, \omega) = \{f \in \mathrm{Diff}\,(M) \mid f^*\omega = \omega\}.$$

Symp(M, ω) is infinite dimensional

- $H : \mathbb{R}^2 \to \mathbb{R}$ is a function of 2 variables.
- Define

$$(X_H)_p = (\frac{\partial H}{\partial y}_p, -(\frac{\partial H}{\partial x})_p),\ p \in \mathbb{R}^2.$$

- X_H is tangent to the level sets $H^{-1}(c)$
- $g_p(t)$ is an integral curve $\mathbb{R}^2$:

$$g_p(0) = p,\ \frac{d}{dt} g_p(t) = (X_H)_{g(t)}$$

$F : \mathbb{R}^2 \to \mathbb{R}^2$ is area preserving

$$F(p) := g_p(1)$$

Follows from the Cartan formula

$$\mathcal{L}_X \omega = i_X d\omega + d(i_X \omega).$$

Conclusion: *Any smooth function H defines F which is area preserving (symplectic, hamiltonian)*

The group $\mathrm{Ham}(M, \omega)$

- $\varphi \in \mathrm{Ham}(M, \omega)$, if $\varphi = \psi_1$, $\psi_t \in \mathrm{Symp}_0(M, \omega)$, $0 \leq t \leq 1$, $\psi_0 = \mathrm{id}$ and
$$i_{X_t}\omega = dH_t, \frac{d}{dt}\psi_t = X_t \circ \psi_t, X_t : M \to TM$$
- $\mathrm{Ham}(M, \omega)$ is a normal subgroup in $\mathrm{Symp}(M, \omega)$,

Motivation:

1. Gromov: $\mathrm{Symp}_0(S^2 \times S^2, \omega_1) \simeq SO(3) \times SO(3)$,
2. Abreu-Anjos, McDuff:

$$\pi_k \, \mathrm{Symp}_0(S^2 \times S^2, \omega_\lambda) \otimes \mathbb{Q} \neq 0$$

   for $\lambda > 1, k = 1, 2, 3, 4m$, $m \in \mathbb{Z}$ is the largest $< \lambda$, and
3. the generator in degree 1 is represented by S^1-action, in degree 3 by 2 actions of $SO(3)$, and the generator in degree $4m$ is "expressed" in terms of the previous ones.

Conclusion and question

- The rational homotopy of $\mathrm{Symp}_0(S^2 \times S^2, \omega_\lambda)$ is determined by the action of compact Lie subgroup $SO(3) \times SO(3)$,
- : Question: *are there analogous statements for other closed* (M, ω) *with symplectic actions of compact Lie groups?*

How much of the rational homotopy of G remains visible in the rational homotopy of $\mathrm{Ham}(M, \omega)$?

We are given:

$$G \to \mathrm{Ham}(M, \omega).$$

When

$$\pi_*(G) \otimes \mathbb{Q} \to \pi_*(\mathrm{Ham}(M, \omega)) \otimes \mathbb{Q}$$

is injective?

Topology of $\mathrm{Symp}(M, \omega)$

Classifying space

- $\mathcal{G}$ - topological group,
- Borel fibration
$$\mathcal{G} \to E\mathcal{G} \to B\mathcal{G} = E\mathcal{G}/\mathcal{G}$$
- classifying space $B\mathcal{G}$
- $\pi_k(\mathcal{G}) = \pi_{k+1}(B\mathcal{G})$

Examples of classifying spaces

- $B\mathbb{Z}^2 = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$,
- Hopf bundle $S^1 \to \mathbb{S}^{2n+1} \to \mathbb{C}P^n$,
$$S^3 \subset S^5 \subset \cdots \subset S^{2n+1} \cdots \subset S^\infty$$
is contractible, thus $BS^1 = \mathbb{C}P^\infty$

Understanding topology of $\mathrm{Ham}(M,\omega)$ via hamiltonian bundles

- Hamiltonian bundles

$$(M,\omega) \to E \to B$$

(the structure group G is a (sub)group $\mathrm{Ham}(M,\omega)$)

- Classifying spaces of G-bundles

$$\begin{array}{ccc} M & \xrightarrow{=} & M \\ \downarrow & & \downarrow \\ E \times_G M & \longrightarrow & M_G = EG \times_G M \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & BG \end{array}$$

Characteristic classes of hamiltonian fiber bundles

$$\begin{tikzcd}
M \arrow[r, "="] \arrow[d] & M \arrow[d] \\
E \arrow[r] \arrow[d] & EG \times_G M \arrow[d] \\
B \arrow[r, "f"] & BG
\end{tikzcd}$$

$G = \mathrm{Ham}(M, \omega)$, $f^* H^*(B\,\mathrm{Ham}(M, \omega)) \subset H^*(B)$

Coupling form and hamiltonian characteristic classes

Objects:

- (M, ω) a closed symplectic manifold of dimension $2n$
- $$(M, \omega) \xrightarrow{\ i\ } E \xrightarrow{\ \pi\ } B$$

  a Hamiltonian fibration over a simply connected base.

Coupling class

A cohomology class $\Omega \in H^2(E)$ uniquely defined by:

$$i^*\Omega = [\omega], \pi_!(\Omega^{n+1}) = 0.$$

It is called the *coupling class*.

The fibre integration is functorial $\implies$ the coupling class is natural $\implies$ the characteristic classes of Hamiltonian fibration:

$$\mu_k(E) = \pi_!(\Omega^{n+k}) \in H^{2k}(B).$$

Fiber integration (for smooth fiber bundles)

$$M \xrightarrow{\quad i \quad} E \xrightarrow{\quad \pi \quad} B$$

$\dim M = n, \dim E = n + k$.

$$\int_E \pi^* \beta \wedge \gamma = \int_B \beta \wedge \pi_! \gamma, \forall \beta \in \Omega^*(B)$$

$$\pi_! d = d\pi_! \implies \pi_! : H^*(E) \to H^{*-\dim M}(B)$$

$\pi_!$ is not a ring homomorphism, but

$$\pi_!(\pi^* \beta \wedge \mu) = \beta \wedge \pi_! \mu$$

Fiber integration (general)

$$M^n \xrightarrow{i} E \xrightarrow{\pi} B$$

The Leray-Serre spectral sequence $E_r^{p,q}$ yields

$$H^{n+k}(E) \xrightarrow{pr} E_\infty^{k,n} \xrightarrow{incl} E_2^{k,n} = H^k(B, H^n(F)) = H^k(B)$$

$$\pi_! = incl \circ pr.$$

Basic questions on $\mu_k(E)$

- Are they non-trivial?
- Are they algebraically independent in the cohomology ring $H^*(B\,\mathrm{Ham}(M,\omega))$?

Results of Reznikov, 1997

If $(M, \omega) = (\mathbb{C}P^n, \omega_{can})$, then μ_k are algebraically independent in $H^*(B\,\mathrm{Ham}(\mathbb{C}P^n, \omega_{can}))$ for $k = 2, ..., n+1$.

Reznikov's conjecture

The hamiltonian characteristic classes μ_k are algebraically independent in the cohomology algebra of the classifying space $B\,\mathrm{Ham}(M_\xi)$ of the coadjoint orbit M_ξ of a compact Lie group.

Here and in the sequel:

- G - a compact Lie group, $\mathfrak{g}$ - the Lie algebra of G,
- $\xi \in \mathfrak{g}^*$ and $M_\xi = G \cdot \xi \subset \mathfrak{g}^*$ is a coadjoint orbit.

A corollary to the Reznikov conjecture

Conjectural theorem

Let G be a compact simple Lie group. For any $\xi \in \mathfrak{g}^*$ the coadjoint orbit M_ξ of ξ satisfies the following. The homomorphism $H^*(B\,\mathrm{Ham}(M_\xi)) \to H^*(BG)$ induced by the action is surjective and its image is generated by the classes μ_k.

Results: Reznikov conjecture is true generically

Theorem 1

Let G be a compact semisimple Lie group and let

$$\mathcal{K} := \{k \in \mathbb{N} \mid \pi_{2k}(BG) \otimes \mathbb{Q} \neq 0\}.$$

There exists a nonempty Zariski open subset $A \subset \mathfrak{g}^*$ in the dual of the Lie algebra of G such that for any $\xi \in A$ the coadjoint orbit M_ξ of ξ satisfies the following. The classes $\mu_k \in H^{2k}(B\,\mathrm{Ham}(M_\xi))$ are algebraically independent for $k \in \mathcal{K}$.

Corollary

Let G be a compact simple Lie group different from $\mathrm{SO}(4k)$ There exists a nonempty Zariski open subset $A \subset \mathfrak{g}^*$ in the dual of the Lie algebra of G such that for any $\xi \in A$ the coadjoint orbit M_ξ of ξ satisfies the following. The homomorphism $H^*(B\,\mathrm{Ham}(M_\xi)) \to H^*(BG)$ induced by the action is surjective and its image is generated by the classes μ_k.

Theorem 1: method of proof

- calculate the characteristic classes μ_k for the universal fibration

$$M_\xi \to BG_\xi = EG \times_G M_\xi \to BG.$$

- Use the fact that the cohomology ring $H^*(BG)$ is a polynomial ring generated by elements with degrees in $\mathcal{K}$. This follows from results about cohomology of classifying spaces and basic rational homotopy theory.

$$H^*(BG) \cong S(\mathfrak{g}^*)^G \cong H^*(BT)^W$$

where $W = W(G, T)$ is the Weyl group.

Proof of Theorem 1: case of flag manifolds

Lemma

Let G be a compact and connected semisimple group. Let ω be a generic homogeneous symplectic form on the flag manifold G/T. Then for a rationally nontrivial homotopy class $f\colon S^{2k} \to BG$ the induced Hamiltonian bundle has a nontrivial class μ_k.

Proof of the Lemma

The pullback diagram for G/T

$$
\begin{array}{ccc}
G/T & \xrightarrow{=} & G/T \\
\downarrow & & \downarrow \\
E & \xrightarrow{\hat{f}} & BT \\
\pi \downarrow & & p \downarrow \\
S^{2k} & \xrightarrow{f} & BG,
\end{array}
$$

Proof of the Lemma: The cohomology of E

For a generator $\sigma \in H^{2k}(S^{2k})$ $\sigma = f^*(\alpha)$ for some $\alpha \in H^{2k}(BG)$, $\implies$ $k > 1$. Since

$$H^*(BT) \cong \mathbb{R}[X_1, ..., X_l], l = rank\ G, |X_i| = 2$$

and

$$\pi^*(\sigma) = \pi^*(f^*(\alpha)) = \hat{f}^*(p^*(\alpha))$$

$\implies$ $H^*(E)$ is generated by degree two classes. Moreover, the inclusion of the fibre induces an isomorphism

$$H^2(E) \cong H^2(G/T).$$

Proof of the Lemma: generic symplectic forms

- $\dim\ G/T = 2n \implies \dim\ E = 2(n+k)$
- an algebraic map $H^2(E) \to H^{2(n+k)}(E) = \mathbb{R}, a \to a^{n+k}$
- $H^2(E) = \langle u_1, ..., u_l \rangle, |u_i| = 2$

$\implies$ there exists a Zariski open subset $Z \subset H^2(E)$

$$Z \subset H^2(E), Z = \{a \in H^2(E) \,|\, a^{n+k} \neq 0\}$$

if there is just one class with nontrivial highest power. Observe that the symmetric map

$$H^2(E)^{\otimes(n+k)} \ni a_1 \otimes \cdots \otimes a_{n+k} \mapsto a_1 \cdot \ldots \cdot a_{k+n} \in H^{n+k}(E)$$

is nontrivial as $H^*(E)$ is generated in dimension 2 and E is closed and oriented. Applying polarization, the map

$$H^2(E) \ni a \mapsto a^{n+k} \in H^{n+k}(E)$$

is nontrivial.

Proof of the Lemma: generic symplectic forms, completion of proof

Choose a G-invariant symplectic form $\omega \in \Omega^2(G/T)$ such that the associated coupling form

$$\Omega \in Z \subset H^2(E).$$

Then $\Omega^{n+k} \neq 0$ and hence

$$\left\langle \mu_k(E), \left[S^{2k}\right] \right\rangle = \left\langle \pi_!(\Omega^{n+k}), \left[S^{2k}\right] \right\rangle = \left\langle \Omega^{n+k}, [E] \right\rangle \neq 0.$$

Corollary 2: algebraic independence of μ_k

Corollary 2

Let $\mathcal{K} = \{k \in \mathbb{N} \,|\, \pi_{2k}(BG) \otimes \mathbb{Q} \neq 0\}$*. For a generic homogeneous symplectic form on a flag manifold* G/T *the classes* $\mu_k \in H^{2k}(B\,\mathrm{Ham}(G/T))$ *are algebraically independent for* $k \in \mathcal{K}$*. Moreover, these classes cannot be generated by classes of smaller degrees.*

Proof of Corollary 2

$$\begin{array}{ccc} G/T & \xrightarrow{=} & G/T \\ \downarrow & & \downarrow \\ E & \longrightarrow & BT \\ \downarrow & & \downarrow \\ S^{2k} & \xrightarrow{f} & BG \end{array}$$

$$\langle \mu_k(E), [S^{2k}] \rangle \neq 0 \implies$$

μ_k cannot be expressed as polynomials of generators of smaller degrees, and

$$H^*(BG) \cong H^*(BT)^W \cong \mathbb{R}[f_1, ..., f_l]$$

Let $A \subset \mathfrak{g}^*$ be defined as

$$A = \{\xi \in \mathfrak{g}^* \mid M_\xi \cong G/T\}, Z \subset H^2(G/T)$$

The intersection of the (Zariski) open and dense subsets $A \times Z$ for each $k \in \mathcal{K}$ has the required properties.

Proof of Theorem 1: general case

$H^*(BG)$

$$H^*(BG) \cong S(\mathfrak{g}^*)^G \cong S(\mathfrak{t}^*)^W$$

for the Weyl group $W = W(G, T)$, $T \subset G$ a maximal torus in G.

Invariant polynomials for M_ξ

Every characteristic class $\mu_k \in H^*(BG)$ for a coadjoint orbit M_ξ, $\xi \in \mathfrak{g}^*$ determines an invariant polynomial $p_{\xi,k} \in S(\mathfrak{g}^*)^G$

1. $X \in \mathfrak{g}$ a fundamental vector field of a circle action on M_ξ;
2. $c_X : BS^1 = \mathbb{C}P^\infty \to BG$ the classifying map;
3. define

$$p_{\xi,k}(X) = \langle c_X^*(\mu_k), \mathbb{C}P^k \rangle \in \mathbb{R}$$

and extend onto $\mathfrak{g}$ by $\mathfrak{g} = \cup_g \operatorname{Ad} g(\mathfrak{t})$.

G-invariance of $p_{\xi,k}$

Formula (Kędra-McDuff, Geom. Top. 2005)

$$p_{\xi,k}(X) = (-1)^k \binom{n+k}{k} \cdot \int_G \langle X, \mathrm{Ad}_g^*(\xi) \rangle^k \, \mathrm{vol}_G \, .$$

Proof of Theorem 1, completion

1. Classes μ_k are algebraically independent in $H^*(BG)$ for a generic ξ defining a flag manifold.
2. An algebraic independence is an open condition
3. $$p_{\xi,k} = (-1)^k \binom{n+k}{k} \cdot \int_G \langle X, \mathrm{Ad}_g^*(\xi) \rangle^k \mathrm{vol}_G$$

   are non-trivial
4. $p_{\xi,k}$ are algebraically independent for a generic $\xi \in \mathfrak{g}^*$ and $k \in \mathcal{K}$
5. $(1) - (4) \implies \mu_k$ as well, because of the formula

   $$p_{\xi,k}(X) = \langle c_X^*(\mu_k), \mathbb{C}P^k \rangle \in \mathbb{R}$$

Example 1: coadjoint orbits of $SU(n)$

Reznikov: the classes μ_k are algebraically independent in $H^*(B\,\mathrm{Ham}(\mathbb{C}P^{n-1}))$ for $k = 2, \dots n \implies$ these classes are also algebraically independent for any coadjoint orbit of $\mathrm{SU}(n)$ which is close to $\mathbb{C}P^{n-1}$.

After the identification of coadjoint and adjoint orbits via the Killing form:

the complex projective space $\mathbb{C}P^{n-1}$ is the adjoint orbit of the diagonal matrix

$$\xi = \mathrm{diag}[-i, -i, \dots, -i, (n-1)i] \in \mathfrak{su}(n).$$

The orbit of an element ξ' from a suitably small neighbourhood of ξ has the same property.

The failure of the algebraic independence - discussion

Proposition 1

Let G be a compact Lie group and let $m \in \mathbb{N}$ be a number for which $\pi_{2m}(BG) \otimes \mathbb{Q} = H^{2m}(BG;\mathbb{Q}) = \mathbb{Q}$. Let $u \in S(\mathfrak{g}^*)^G$ be a nontrivial invariant polynomial of degree m. The class $\mu_m \in H^{2m}(BG)$ is trivial for the coadjoint orbit M_ξ if and only if $u(\xi) = 0$.

Proof of Proposition 1

Assumptions +

$$H^*(BG) = S(\mathfrak{g}^*)^G$$

$$\Longrightarrow$$

u is unique up to a constant.

Example:

$$H^*(BSU(n)) = \mathbb{Q}[u_2, ..., u_n],\ |u_i| = 2i$$

Forgetting the grading we get

$$H^*(BSU(n)) \cong S(\mathfrak{su}(n)^*)^{SU(n)}$$

and generators u_i correspond to the invariant polynomials of degrees i, which are defined uniquely up to a constant.

Proof of Proposition 1, continued

$p_{\xi,m}(X)$ can be considered as a bi-invariant polynomial on $\mathfrak{g} \otimes \mathfrak{g}^* \implies$ there exists a degree m invariant polynomial v on $\mathfrak{g}$ such that

$$p_{\xi,m}(X) = u(\xi) \cdot v(X).$$

$p_{\xi,m}(-)$ is nontrivial for a generic $\xi \implies v$ is nonzero. Hence $p_{\xi,m}(X)$ is trivial if and only if $u(\xi) = 0$.

A counterexample to the Reznikov conjecture

Proposition 2

The class $\mu_3 \in H^6(B\mathrm{SU}(n))$ *is trivial for the adjoint orbit of the diagonal matrix* $\mathrm{diag}[X_1, \dots, X_n] \in \mathfrak{su}(n)$ *if and only if* $\sum X_i^3 = 0$*. In particular, the class* μ_3 *is trivial for the grassmannian* $G(m, 2m)$ *of m-planes in* $\mathbb{C}^{2m}$*.*

Explicit example

$$G(m, 2m) = SU(2m) \cdot X, X = \mathrm{diag}[i, ..., i, -i, ..., -i]$$

This follows since $S(\mathfrak{su}(n))^{SU(n)}$ is generated by polynomials of the form

$$X \to \sum X_i^k$$

where $X_i \in \mathbb{C}$ are the eigenvalues of the matrix X. Thus, any invariant polynomial of degree 3 is equal to $\sum X_i^3$ (up to a constant). By Proposition 2 and the fact that X is a generic zero of $\sum X_i^3$, we get $u(\xi) = 0$ for ξ, the dual of X, and, therefore, $\mu_3 = 0$ for $M_\xi = G(m, 2m)$.

Zariski open subsets in $\mathfrak{g}^*$

1. $$A = \{\xi \in \mathfrak{t}^* \mid M_\xi \cong G/T\}$$
2. C consistst of $\xi \in \mathfrak{t}^*$ such that μ_k asso.with M_ξ are algebraically independent.

A and C do not contain each other.

Example

$$SO(2n)/U(n) \to E \to S^{2n}$$

$$\mu_n \neq 0$$

$$\Longrightarrow$$

μ_n is non-trivial for $SO(2n)/H$, where

$$H = U(n_1) \times ... \times U(n_k), n_1 + \cdots n_k = n.$$

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