Symplectic varieties with invariant Lagrangian subvarieties

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Main Thesis

We consider symplectic algebraic varieties equipped with a Hamiltonian reductive group action which contain an invariant Lagrangian subvariety.

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Outline

Symplectic geometry

Let G be a reductive group, B be a Borel subgroup, X be an algebraic G-variety.

Definition

The *complexity* $c(X)$ is the codimension of general B-orbits in X.

It can also be defined as the minimal codimension of B-orbits in X and, by the Rosenlicht theorem, coincides with the transcendence degree (over $\Bbbk$) of the field $\Bbbk(X)^B$ of B-invariant rational functions.

Definition

The *weight lattice* of X is the set $\Lambda(X)$ of eigenweights of all (nonzero) B-semi-invariant rational functions on X.

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The *rank* of X is $r(X) = \operatorname{rk} \Lambda(X)$.

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Our work was motivated by the following result of Panyushev (1999).

Theorem

Let X be a smooth G-variety and $Y \subset X$ be a smooth G-subvariety. Denote by $N = N(X/Y)$ and $N^* = N^*(X/Y)$ the normal and conormal bundle of Y in X, respectively. Then $c(X) = c(N) = c(N^*)$ and $r(X) = r(N) = r(N^*)$.

Note: N^* is a Lagrangian subvariety T^*X.

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Symplectic geometry

- (M, ω), symplectic manifold (over $\mathbb{K} = \mathbb{R}, \mathbb{C}$) or
  smooth algebraic variety (over $\mathbb{K} = \mathbb{C}$),

  ω is a nondegenerate closed 2-form on M;
- ∇f, skew gradient of $f : M \supset U \to \mathbb{K}$,
  $df(v) = \omega(\nabla f, v)$, $\forall v \in TM$;
- $\{f, g\} = \omega(\nabla f, \nabla g)$, Poisson bracket.

A submanifold / smooth algebraic subvariety $S \subseteq M$ is:

- *isotropic* if $\omega|_{T_pS} = 0$, $\forall p \in S$;
- *coisotropic* if $\omega|_{(T_pS)^\angle} = 0$, $\forall p \in S$;
- *Lagrangian* = isotropic + coisotropic.

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Hamiltonian actions

Lie/algebraic group action $G \curvearrowright M$ is *Hamiltonian* if:

- it preserves ω;
- $\exists$ *moment map* $\Phi : M \to \mathfrak{g}^*$:
  - Φ is G-equivariant;
  - $\nabla(\Phi^*\xi) = \xi_*$, $\forall \xi \in \mathfrak{g}$;
  - $\langle d_p\Phi(v), \xi \rangle = \omega(\xi p, v)$, $\forall p \in M$, $v \in T_pM$;

  Notation: $\xi_*(p) = \xi p = \frac{d}{dt}\big|_{t=0} \exp(t\xi)p$: velocity vector
- $\{\Phi^*\xi, \Phi^*\eta\} = \Phi^*([\xi, \eta])$, $\forall \xi, \eta \in \mathfrak{g}$.

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Basic example: cotangent bundles

Example

$M = T^*X$, $\omega = \sum_i dx_i \wedge dy_i$,
x_i are local coordinates on X, y_i are dual coordinates in $T^*_x X$.

$G \curvearrowright X$ induces Hamiltonian action $G \curvearrowright T^*X$,
$\langle \Phi(p), \xi \rangle = \langle p, \xi x \rangle$, $\forall x \in X$, $p \in T^*_x X$, $\xi \in \mathfrak{g}$.

Zero section $S \subset T^*X$ is Lagrangian.

Conormal bundles $N^*(X/Y) = \{p \in T^*_x X \mid x \in Y,\ \langle p, T_x Y \rangle = 0\}$ are Lagrangian for any $Y \subseteq X$.

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Structure of a neighborhood of a Lagrangian submanifold

Assume $S \subset M$ Lagrangian.

Darboux–Weinstein Theorem $\implies$ $M \simeq T^*S$ (C^∞ symplectomorphism) in a neighborhood of S

G compact Lie group, $G \curvearrowright M$ Hamiltonian, S G-stable $\implies$ G-equivariant local symplectomorphism $M \simeq T^*S$ (B. Kostant)

G reductive algebraic group, M Hamiltonian G-variety, $S \subset M$ G-stable Lagrangian subvariety: G-equivariant local symplectomorphism may not exist.

Obstruction: the structure of isotropy representations.

- $G_p \curvearrowright T_p(T^*S) = T_pS \oplus T_p^*S$ splits.
- May happen that T_pS has no G_p-stable complement in T_pM.

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Example: complete conics

Example

$\mathbb{P}^5 = \mathbb{P}(\mathrm{Sym}_{3\times 3}(\mathbb{C}))$, space of conics in $\mathbb{P}^2$
$F \subset \mathbb{P}^5$, set of double lines
$X = \mathrm{Bl}_F(\mathbb{P}^5)$, variety of *complete conics*
$G = SL_3(\mathbb{C}) \curvearrowright X \supset Y$, the unique closed orbit

Put $M = T^*X$, $S = N^*(X/Y)$.

$\exists$ unique $y \in Y$ such that $G_y = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\}$

$p \in S_y$ general point $\implies G_p^\circ = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}$

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Setup

From now on:

M is an irreducible symplectic algebraic variety;
G is a connected reductive algebraic group, $\mathfrak{g} = \operatorname{Lie} G$;
$G \curvearrowright M$ is a Hamiltonian action;
$\Phi : M \to \mathfrak{g}^*$ is the moment map.

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Invariants of a Hamiltonian action

Definition

Corank $\operatorname{cork} M = \operatorname{rk} \omega|_{(\mathfrak{g}p)^\angle}$;
Defect $\operatorname{def} M = \dim \mathfrak{g}p \cap (\mathfrak{g}p)^\angle$ $\qquad$ ($p \in M$ general point).

Properties:

1. $\operatorname{Ker} d_p\Phi = (\mathfrak{g}p)^\angle$;
2. $\operatorname{Im} d_p\Phi = (\mathfrak{g}_p)^\perp$;
3. $\dim \overline{\Phi(M)} = \dim Gp$ (p general) $\Longleftarrow$ (2);
4. $\operatorname{def} M = \dim \overline{\Phi(M)}/G \Longleftarrow$ (3), (1);
5. $\operatorname{cork} M = \dim M - \dim \overline{\Phi(M)} - \dim \overline{\Phi(M)}/G \Longleftarrow$ (6), (3), (4);
6. $\operatorname{cork} M + \operatorname{def} M = \dim M/G$.

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Invariants of a Hamiltonian action

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How to formulate an analog of Panyushev's theorem for Hamiltonian varieties?

Answer:

Theorem (F.Knop 91)

$2c(X) = \operatorname{cork} T^*X, \quad r(X) = \operatorname{def} T^*X.$

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Let M be a Hamiltonian G-variety and let $S \subset M$ be an irreducible G-stable Lagrangian subvariety. Then $2c(S) = \operatorname{cork} M, \quad r(S) = \operatorname{def} M.$

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Ideas of the proof

1. Deformation to the normal bundle

   $\exists$ flat family $\widehat{M} \to \mathbb{A}^1$ with fibers
   $M_c \simeq M$, $\forall c \neq 0$,
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2. Foliation of horospheres

   Horosphere = orbit of a (fixed) maximal unipotent subgroup $U \subset G$

   Suppose S is quasiaffine. Denote:
   $\mathcal{U} \subset T^*S$, conormal bundle to foliation of generic horospheres in S;
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   $P_0 \subset G$, normalizer of a generic horosphere.

Theorem ([Knop, 1994])

$$\overline{\Phi(\mathcal{U})} = \mathfrak{p}_0^\perp, \qquad \overline{G\mathcal{U}} = T^*S, \qquad \overline{\Phi(T^*S)} = G\mathfrak{p}_0^\perp.$$

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Hamiltonian structure on the deformation to the normal bundle.
Consider the blow up of $S \times \{0\}$ in $M \times \mathbb{A}^1$. The exceptional divisor is isomorphic to the projective bundle $\mathbb{P}(N \oplus \Bbbk)$ over $S \times \{0\}$, where N is the normal bundle of $S \subset M$. The strict preimage $\check{M}$ of $M \times \{0\}$ is nothing else but the blowup of $M \times \{0\}$ at $S \times \{0\}$. These two divisors intersect in $\mathbb{P}(N)$, the exceptional divisor of $\check{M} \to M$. Removing $\check{M}$ we obtain a smooth variety $\widehat{M}$ together with a smooth morphism $\delta : \widehat{M} \to \mathbb{A}^1$ such that $\delta^{-1}(\mathbb{A}^1 \setminus \{0\}) \simeq M \times (\mathbb{A}^1 \setminus \{0\})$ and $\delta^{-1}(0) \simeq N$.

In more algebraic terms,

$$\varphi : \widehat{M} \to M \times \mathbb{A}^1 \to M \text{ is an affine morphism.}$$

$$\varphi_* \mathcal{O}_{\widehat{M}} = \bigoplus_{n=-\infty}^{\infty} \mathcal{I}_S^n t^{-n} \subset \mathcal{O}_M[t^{\pm 1}],$$

where t is the coordinate on $\mathbb{A}^1$, $\mathcal{I}_S \lhd \mathcal{O}_M$ is the ideal sheaf defining S, and $\mathcal{I}_S^{-1} = \mathcal{I}_S^{-2} = \cdots = \mathcal{O}_M$ by definition.

Ideas of the proof

Lemma

Let $S \subset M$ is a coisotropic subvariety and $\mathcal{I}_S$ be the sheaf of ideals defining S. Then $\{\mathcal{I}_S^n, \mathcal{I}_S^m\} \subset \mathcal{I}_S^{n+m-1}$.

Since the subvariety $S \subset M$ is coisotropic (which means that $TS \supset (TS)^{\angle}$ in $TM|_S$), the skew gradients of $f \in \mathcal{I}_S$ are tangent to S, i.e., $\langle d\mathcal{I}_S, \nabla\mathcal{I}_S \rangle = 0$ on S or, equivalently, $\{\mathcal{I}_S, \mathcal{I}_S\} \subset \mathcal{I}_S$.

Now the Poisson bracket on $\varphi_*\mathcal{O}_{\widehat{M}}$ is defined as $\{ft^{-n}, gt^{-m}\} = \{f, g\}t^{-n-m+1}$, $\forall f \in \mathcal{I}_S^n$, $g \in \mathcal{I}_S^m$.

If S is Lagrangian subvariety we define the *total moment map* $\widehat{\Phi} : \widehat{M} \to \mathfrak{g}^* \times \mathbb{A}^1$ such that the dual algebra homomorphism $\widehat{\Phi}^* : \Bbbk[\mathfrak{g}^*][t] \to \Bbbk[\widehat{M}]$ is defined by the formulæ: $\widehat{\Phi}^*\xi = \Phi^*\xi \cdot t^{-1}$, $\forall \xi \in \mathfrak{g}$, and $\widehat{\Phi}^*t = t$. Which is well defined since $\Phi^*\xi \subset \mathcal{I}_S$.

Ideas of the proof

$T^*S \rightsquigarrow M, \quad \mathcal{U} \rightsquigarrow \mathcal{W}$

Construction of $\mathcal{W}$:

- Choose P_0-invariant functions $F_1, \dots, F_m : M \supset \mathring{M} \to \mathbb{C}$ such that $dF_1|_S, \dots, dF_m|_S$ span $\mathcal{U}$.
- Spread S along the trajectories of $\nabla F_1, \dots, \nabla F_m$.

Proposition

$$\overline{\Phi(\mathcal{W})} = \mathfrak{p}_0^\perp, \qquad \overline{G\mathcal{W}} = M, \qquad \overline{\Phi(M)} = G\mathfrak{p}_0^\perp.$$

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Coisotropic case

Let now $S \subset M$ be coisotropic G-stable.
Assume:

$$\mathfrak{g}x \subset (T_x S)^{\angle}, \qquad \forall x \in S \qquad (\diamondsuit)$$

Theorem

If $(\diamondsuit)$ *holds, then*

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Is it possible to ~~waive~~ weaken $(\diamondsuit)$*?*

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Let H be an algebraic group, $\mathfrak{h} = \operatorname{Lie} H$.

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Conjecture (Elashvili)

$$\forall x \in \mathfrak{g}^* \simeq \mathfrak{g} : \quad \operatorname{ind}\mathfrak{g}_x = \operatorname{ind}\mathfrak{g}$$

- reduced to nilpotent x;
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Natural action $G \times G \curvearrowright G$ (by left/right multiplication) yields Hamiltonian action $G \times G \curvearrowright M = T^*G \simeq G \times \mathfrak{g}^*$.

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Equality of dim's of LHS's would imply $\operatorname{ind} \mathfrak{g} = \operatorname{ind} \mathfrak{g}_x$.
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