Constructing n-Engel Lie rings

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Outline

Lie rings

A **Lie ring** L is a $\mathbb{Z}$-module equipped with a multiplication

$$\begin{aligned} [\ ,\ ] : L \times L &\longrightarrow L \\ (x, y) &\longmapsto [x, y] \end{aligned}$$

such that, for all x, y, z in L

- $[x, x] = 0$;
- $[x, y] + [y, x] = 0$;
- $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$, (**Jacobi identity**).

Free magma

Let X be a finite set of symbols. The **free magma** on X is the set $M(X)$ defined as:

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- if $m, n \in M(X)$, then also the pair $(m, n) \in M(X)$.

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For $m \in M(X)$ we define its **degree** recursively:

- $\deg(m) = 1$ if $m \in X$;
- $\deg(m) = \deg(m') + \deg(m'')$ if $m = (m', m'')$.

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We use a total and multiplicative **order** $<$ on $M(X)$.

Free non-associative rings

Let $A_{\mathbb{Z}}(X)$ the $\mathbb{Z}$-span of $M(X)$.
We extend the binary operation $\cdot$ on $M(X)$ bilinearly to $A_{\mathbb{Z}}(X)$, then $A_{\mathbb{Z}}(X)$ becomes a non-associative ring called the **free non-associative ring** over $\mathbb{Z}$ on X.

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The elements of $M(X)$ that occur in a $f \in A_{\mathbb{Z}}(X)$ are called **monomials** of f. The **leading monomial** of f is denoted by $\mathrm{LM}(f)$ and its coefficient by $\mathrm{LC}(f)$. The **degree** of f will be the degree of $\mathrm{LM}(f)$.

Free Lie rings

Let I_0 be the ideal of $A_{\mathbb{Z}}(X)$ generated by all (m, m), $(m, n) + (n, m)$ and $(m, (n, p)) + (n, (p, m)) + (p, (m, n))$, for $m, n, p \in M(X)$. Let $L(X) = A_{\mathbb{Z}}(X)/I_0$.
Then $L(X)$ is a Lie ring called the **free Lie ring** on X.

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We say that a Lie ring defined in this way is given by a **finite presentation** and the Lie ring is said to be **finitely presented**.

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For all $x_i, x_j \in \mathfrak{B}$, there are n^3 **structure constants** $c_{ij}^k \in \mathbb{Z}$, such that

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$$[x_i, x_j] = \sum_{k=1}^{n} c_{ij}^k x_k.$$

Also, L, as an abelian group, is isomorphic to

$$\mathbb{Z}/n_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/n_r\mathbb{Z} \oplus \mathbb{Z}^{n-r},$$

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where $n_1, \ldots, n_r$ are **invariant factors** such that n_i divides n_{i+1}.

We call the set of structure constants c_{ij}^k together with the set of invariant factors $n_1, \ldots, n_r$, a **multiplication table** of L.

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The best we can hope is to have an algorithm that constructs a multiplication table for a finitely presented Lie ring L whenever L happens to be finite-dimensional, that is finitely generated as an abelian group.

Product prescriptions

Let $\sigma = (m_1, \ldots, m_k)$ be a sequence of elements of $M(X)$ and let $\delta = (d_1, \ldots, d_k)$ be a sequence of letters $d_i \in \{l, r\}$. We call the pair $\alpha = (\sigma, \delta)$ a **product prescription**.

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Corresponding to α there is a map $P_\alpha : M(X) \to M(X)$ defined as:

- If $k = 0$ then $P_\alpha(m) = m$ for all m.
- If $k > 0$ we set $\beta = ((m_2, \ldots, m_k), (d_2, \ldots, d_k))$ and

$$P_\alpha(m) = \begin{cases} P_\beta((m_1, m)), & \text{if } d_1 = l; \\ P_\beta((m, m_1)), & \text{if } d_1 = r. \end{cases}$$

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An $m \in M(X)$ is said to be a **factor** of $n \in M(X)$ if there is a product prescription α such that $P_\alpha(m) = n$.

Gröbner bases in $A_{\mathbb{Z}}(X)$

Let $G \subset A_{\mathbb{Z}}(X)$ be a finite set and let $f \in A_{\mathbb{Z}}(X)$.
Let $g_1, \ldots, g_s \in G$ be all elements of G such that $\mathrm{LM}(g_i)$ is a factor of $\mathrm{LM}(f)$.

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Suppose that $\mathrm{LC}(f)$ is divisible by $d = \gcd(c_1, \ldots, c_s)$, where $c_i = \mathrm{LC}(g_i)$. Let $e_i > 0$ be such that $e_1 c_1 + \ldots + e_s c_s = d$.

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We say that f **reduces** modulo G to f' where

$$f' = f - c(e_1 P_{\alpha_1}(g_1) + \ldots + e_s P_{\alpha_s}(g_s)).$$

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Let $J \subset A_{\mathbb{Z}}(X)$ be an ideal. We call a $G \subset J$ a **Gröbner basis** of J if every $f \in J$ reduces to zero modulo G.

The algorithm `FpLieRing`

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- (m, m) for $m \in M(X)$,
- $(m, n) + (n, m)$ for $m, n \in M(X)$,
- $\mathrm{Jac}(m, n, p)$ for $m, n, p \in M(X)$, where

$$\mathrm{Jac}(m, n, p) = (m, (n, p)) + (n, (p, m)) + (p, (m, n)),$$

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Then $L \cong A_{\mathbb{Z}}(X)/J$.

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- the set of **monic elements** G^{mon} will be self-reduced then it is a Gröbner basis;
- the set of **non-monic elements** B is closed under multiplication (that means if $b \in B$ and x is a generator then (x, b) lies in the span of B).

Example

Let $X = \{x, y\}$, with $x < y$, and $R = \{h_1, h_2, h_3\}$, where

$$\begin{aligned} h_1 &= [x, [x, y]] + [x, y], \\ h_2 &= 3[y, [y, [x, y]]] + 6[y, [x, y]] + 2y, \\ h_3 &= [y, [y, [y, [x, y]]]] + 2[y, [y, [x, y]]]. \end{aligned}$$

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Using the algorithm `FpLieRing`, we can calculate a **Gröbner basis** G of the ideal J of $A_{\mathbb{Z}}(X)$, generated by (m,m), $(m,n)+(n,m)$, $\mathrm{Jac}(m,n,p)$, for $m,n,p \in M(X)$, and the elements of R.

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Also we want to determine **the set $\mathfrak{B}$ of normal monomials** modulo G.

Example

- The only relation of **degree** ≤ 3 is, by h_1,

$$g_1 = (x, (x, y)) + (x, y),$$

then we have

$$G_3 = \{g_1\}, \quad B_3 = \emptyset \quad \text{and} \quad M_{\leq 3} = \{x, y, (x, y), (y, (x, y))\}.$$

Example

- In **degree** 4, by h_2, we take

$$g_2 = 3(y,(y,(x,y))) + 6(y,(x,y)) + 2y.$$

Also we have

$$\mathrm{Jac}(x,y,(x,y)) = (x,(y,(x,y))) - (y,(x,(x,y))),$$

and because we can reduce that modulo g_1, we obtain the new relation

$$g_3 = (x,(y,(x,y))) + (y,(x,y)).$$

Then

$$G_4 = \{g_1, g_2, g_3\}, \quad B_4 = \{g_2\} \quad \text{and} \quad M_4 = \{(y,(y,(x,y)))\}.$$

Example

- In **degree** 5, by h_3, we put

$$g_4 = (y,(y,(y,(x,y)))) + 2(y,(y,(x,y))).$$

Also, $\mathrm{Jac}(x,y,(y,(x,y)))$ reduce to

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The first implies the new relation

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$G_5 = \{g_1, \dots, g_6\}$, $B_5 = \{g_2, g_6\}$ and $M_5 = \{(x,(y,(y,(x,y))))\}$.

Example

If we proceed in this way **we obtain** that

$$G = \{f_1, f_2, f_3, f_4, f_5, f_6, f_7\} \quad \text{and} \quad B = \{f_1, f_2, f_4\},$$

where

$$\begin{aligned}
f_1 &= 8y,\\
f_2 &= 4(x,y) + 4y,\\
f_3 &= (x,(x,y)) + (x,y),\\
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f_5 &= (x,(y,(x,y))) + (y,(x,y)),\\
f_6 &= (y,(y,(x,y))) + 2(y,(x,y)) + 6y,\\
f_7 &= ((x,y),(y,(x,y))) + 6(x,y) + 6y.
\end{aligned}$$

Example

The **set of normal monomials** modulo G is then $\mathfrak{B} = \{e_1, e_2, e_3, e_4\}$ with relations

$$4e_1 = 0,$$
$$4e_2 + 4e_3 = 0,$$
$$8e_3 = 0,$$

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We want to calculate a **multiplication table** of the Lie ring L.

Example

The set $\mathfrak{B}$ forms also a **basis of the non-associative ring** $\mathcal{A} = A_{\mathbb{Z}}(X)/\widetilde{J}$, where $\widetilde{J}$ is the ideal of $A_{\mathbb{Z}}(X)$ generated by G^{mon}.

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We can prove that $\{e_1, e_2 + e_3, e_3, e_4\}$ is a **basis of $\mathcal{A}$** such that

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There is a **homomorphism** σ of $\mathcal{A}$ onto $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}$ taking $\alpha_1 e_1 + \alpha_2(e_2 + e_3) + \alpha_3 e_3 + \alpha_4 e_4$, where $\alpha_1, \dots, \alpha_4 \in \mathbb{Z}$, to

$$(\alpha_1 \bmod 4, \alpha_2 \bmod 4, \alpha_3 \bmod 8, \alpha_4).$$

Example

Let now $v_1, \ldots, v_4$ such that

$$
\begin{aligned}
v_1 &= \sigma(e_1), \\
v_2 &= \sigma(e_2 + e_3), \\
v_3 &= \sigma(e_3), \\
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Then $\{v_1, v_2, v_3, v_4\}$ is a **basis** of L with $4v_1 = 0$, $4v_2 = 0$ and $8v_3 = 0$.

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Then $\{v_1, v_2, v_3, v_4\}$ is a **basis** of L with $4v_1 = 0$, $4v_2 = 0$ and $8v_3 = 0$.

We can calculate **all products** $[v_i, v_j]$, for all $i, j = 1, \ldots, 4$, $i < j$, by means of

$$[v_i, v_j] = \sigma(\sigma^{-1}(v_i) \cdot \sigma^{-1}(v_j)).$$

Example

The **multiplication table** of L is

$$
\begin{aligned}
[v_1, v_2] &= 2v_1 + 2v_2 + 6v_3,\\
[v_1, v_3] &= 2v_1 + 6v_3,\\
[v_1, v_4] &= v_1,\\
[v_2, v_3] &= 3v_1,\\
[v_2, v_4] &= 0,\\
[v_3, v_4] &= 3v_2 + v_3,\\
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- `FpLieRing` **will terminate** whenever the input defines a finite-dimensional Lie ring. Otherwise it will **run forever**.

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- `FpLieRing` is **similar** to known algorithms where Gröbner bases are used to construct finitely presented **Lie algebras**. In our we extend these methods to deal with finitely presented Lie rings. The fact that **we work over $\mathbb{Z}$** and not over a field causes a lot of **additional problems**.
- It is also possible that a finitely presented Lie ring is defined by an **infinite** set of relations. `FpLieRing` can deal with this provided that we can only have a finite number of relations of a given degree.

The algorithm LieNQ

In a paper of 1997, **C. Schneider** described an algorithm, called `LieNQ`, to compute so-called **nilpotent quotients** of finitely presented Lie rings.

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When the input relations are **homogeneous** (i.e., each relation has monomials of the same degree), then it is possible to reformulate the algorithm `FpLieRing` in such a way that it becomes **very similar** to Schneider's algorithm. So for this case the two approaches yield similar algorithms.

The algorithm `LieNQ`

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However, the approach via Gröbner bases leads to a **more general** algorithm, that will work whenever the finitely presented Lie ring is finite-dimensional.

Preliminars

We have applied the algorithm `FpLieRing` to construct the **biggest Lie ring** that is

- generated by t elements;
- satisfies the n-Engel condition

for various t and n.

The *n*-Engel condition

A Lie ring L satisfies the ***n*-Engel condition** if

$$[\underbrace{x, [x, [\ldots, [x}_{n}, y] \ldots]]] = 0$$

for all x, y in L.

The n-Engel condition

A Lie ring L satisfies the ***n*-Engel condition** if

$$\underbrace{[x,[x,[\ldots,[x}_{n},y]\ldots]]]=0$$

for all x, y in L.

We will use the **right normed convention** for iterated commutators.
For example, $[xxxxy]$ will be the element $[x[x[x[xy]]]]$ of L.

Lower central series

The **lower central series** of L is defined as:

- $L^1 = L$;
- $L^{r+1} = [L, L^r]$, for $r \geq 1$;

where $[L, L^r]$ is the subring of L generated (as an abelian group) by all $[x, y]$ for $x \in L$ and $y \in L^r$.
L is **nilpotent** if $L^{s+1} = 0$ for some s, and the smallest such s is called the **nilpotency class**.

$E(t, n)$

In 1989 **Zelmanov** shows that: *a finitely-generated Lie ring that satisfies an n-Engel condition is nilpotent*.

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What is the structure of $E(t, n)$?

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Let $E(t, n)$ be the *biggest* Lie ring with t generators which satisfies the n-Engel condition.

What is the structure of $E(t, n)$?

Higgins and Traustason have studied the structure of $E(t, n)$ *over fields*.

Results over fields

Let L be an algebra over a field k.
In 1953 **Higgins** shows that:

- 2-Engel condition implies $L^4 = 0$;
- 3-Engel condition implies

$$L^7 = 0 \qquad \text{if char } k \neq 2, 5.$$

In 1993 **Traustason** shows that:

- 3-Engel condition implies

$$L^5 = 0 \qquad \text{if char } k \neq 2, 5;$$

- 4-Engel condition implies

$$L^c = 0, \quad c < 9 \qquad \text{if char } k \neq 2, 3, 5.$$

Main problem

We have studied the structure of $E(t, n)$ over $\mathbb{Z}$ rather than over a field. For this, we have applied our algorithm for various t and n.

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One problem when dealing with the n-Engel condition is:

The condition $[x \dots xy] = 0$ is not multilinear.

In fact, it is only linear in y.

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In order to establish whether a Lie ring L is n-Engel it is *not* sufficient to check this condition for the elements of a basis.

Main problem

We have studied the structure of $E(t, n)$ over $\mathbb{Z}$ rather than over a field. For this, we have applied our algorithm for various t and n.

One problem when dealing with the n-Engel condition is:

The condition $[x \dots xy] = 0$ is not multilinear.

In fact, it is only linear in y.

In order to establish whether a Lie ring L is n-Engel it is *not* sufficient to check this condition for the elements of a basis.

Let L be generated as an abelian group by $\mathfrak{B} = \{x_1, \dots, x_m\}$. We have determined several **sets of conditions** on the elements of $\mathfrak{B}$ only, that are necessary and sufficient for L to satisfy the n-Engel condition.

2-Engel Lie rings

Let L be generated as an abelian group by $\mathfrak{B} = \{x_1, \ldots, x_m\}$.
The **2-Engel condition** is $[xxy] = 0$ for all $x, y \in L$.
Then $[x_i x_i y] = 0$ for all $x_i \in \mathfrak{B}$ and $y \in L$.
Let $x = x_i + p_j x_j$ with $p_j = \pm 1$. We have

$$0 = [(x_i + p_j x_j)(x_i + p_j x_j)y] = [x_i x_i y] + p_j([x_i x_j y] + [x_j x_i y]) + [x_j x_j y]$$

then

$$[x_i x_j y] + [x_j x_i y] = 0 \quad \forall x_i, x_j \in \mathfrak{B}, \; i < j, \quad y \in L.$$

Let $x = p_{j_1} x_{j_1} + \ldots + p_{j_s} x_{j_s}$, $x_{j_r} \in \mathfrak{B}$ and $p_{j_r} = \pm 1$. We have

$$0 = [xxy] = \sum_r p_{j_r}^2 [x_{j_r} x_{j_r} y] + \sum_{r \neq t} p_{j_r} p_{j_t} ([x_{j_r} x_{j_t} y] + [x_{j_t} x_{j_r} y]).$$

2-Engel Lie rings

A Lie ring L is **2-Engel** if and only if

- $[x_i x_i y] = 0$;
- $[x_i x_j y] + [x_j x_i y] = 0$;

for all $y \in L$, $x_i, x_j \in \mathfrak{B}$ and $i < j$.

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We have showed that $E(t, 2)$ has dimension:

$$\dim(E(t,2)) = t + \binom{t}{2} + \binom{t}{3}$$

and nilpotency class 3, for $t > 2$.

3-Engel Lie rings

The **3-Engel condition** is $[xxxy] = 0$ for all $x, y \in L$. Then $[x_i x_i x_i y] = 0$ for all $x_i \in \mathfrak{B}$ and $y \in L$.
If $x = x_i + p_j x_j$ with $p_j = \pm 1$

$$\begin{aligned} 0 &= [x_i x_i x_j y] + [x_i x_j x_i y] + [x_j x_i x_i y] \pm ([x_i x_j x_j y] + [x_j x_i x_j y] + [x_j x_j x_i y]). \\ & [(x_i^{(2)} x_j)^* y] = [x_i x_i x_j y] + [x_i x_j x_i y] + [x_j x_i x_i y] \\ & [(x_i x_j^{(2)})^* y] = [x_i x_j x_j y] + [x_j x_i x_j y] + [x_j x_j x_i y] \\ & [(x_i^{(2)} x_j)^* y] \pm [(x_i x_j^{(2)})^* y] = 0 \end{aligned}$$

$\forall x_i, x_j \in \mathfrak{B},\ i < j,\ y \in L.$
If $x = x_i + p_j x_j + p_k x_k$ with $p_j, p_k = \pm 1$

$$[(x_i x_j x_k)^* y] = 0 \quad \forall x_i, x_j, x_k \in \mathfrak{B},\ i \leq j \leq k, \quad y \in L.$$

3-Engel Lie rings

A Lie ring L is **3-Engel** if and only if

- $[(x_i^{(3)})^* y] = 0$
- $[(x_i^{(2)} x_j)^* y] + [(x_i x_j^{(2)})^* y] = 0$
- $[(x_i^{(2)} x_j)^* y] - [(x_i x_j^{(2)})^* y] = 0$
- $[(x_i x_j x_k)^* y] = 0$

for all $y \in L$, $x_i, x_j, x_k \in \mathfrak{B}$ and $i \leq j \leq k$.

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for all $y \in L$, $x_i, x_j, x_k \in \mathfrak{B}$ and $i \leq j \leq k$.
If in the third relation

$$[x_i x_j x_k y] + [x_i x_k x_j y] + [x_j x_i x_k y] + [x_j x_k x_i y] + [x_k x_i x_j y] + [x_k x_j x_i y] = 0$$

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we put $i \leq j = k$ we obtain

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- $[(x_i x_j x_k)^* y] = 0$

for all $y \in L$, $x_i, x_j, x_k \in \mathfrak{B}$ and $i \leq j \leq k$.

4-Engel Lie rings

A Lie ring L is **4-Engel** if and only if

- $[(x_i^{(4)})^* y] = 0;$
- $[(x_i^{(3)} x_j)^* y] + [(x_i^{(2)} x_j^{(2)})^* y] + [(x_i x_j^{(3)})^* y] = 0;$
- $[(x_i^{(3)} x_j)^* y] - [(x_i^{(2)} x_j^{(2)})^* y] + [(x_i x_j^{(3)})^* y] = 0;$
- $[(x_i^{(2)} x_j x_k)^* y] + [(x_i x_j^{(2)} x_k)^* y] + [(x_i x_j x_k^{(2)})^* y] = 0;$
- $[(x_i^{(2)} x_j x_k)^* y] - [(x_i x_j^{(2)} x_k)^* y] + [(x_i x_j x_k^{(2)})^* y] = 0;$
- $[(x_i^{(2)} x_j x_k)^* y] + [(x_i x_j^{(2)} x_k)^* y] - [(x_i x_j x_k^{(2)})^* y] = 0;$
- $[(x_i^{(2)} x_j x_k)^* y] - [(x_i x_j^{(2)} x_k)^* y] - [(x_i x_j x_k^{(2)})^* y] = 0;$
- $[(x_i x_j x_k x_r)^* y] = 0;$

for all $y \in L$, $x_i, x_j, x_k, x_r \in \mathfrak{B}$ and $i \leq j \leq k \leq r$.

4-Engel Lie rings

A Lie ring L is **4-Engel** if and only if

(1) $[(x_i^{(4)})^* y] = 0;$

(2) $[(x_i^{(3)} x_j)^* y] + [(x_i^{(2)} x_j^{(2)})^* y] + [(x_i x_j^{(3)})^* y] = 0;$

(3) $[(x_i^{(2)} x_j x_k)^* y] + [(x_i x_j^{(2)} x_k)^* y] + [(x_i x_j x_k^{(2)})^* y] = 0;$

(4) $[(x_i x_j x_k x_r)^* y] = 0;$

(5) $2[(x_i^{(2)} x_j^{(2)})^* y] = 0;$

(6) $2[(x_i x_j^{(2)} x_k)^* y] = 0;$

(7) $2[(x_i x_j x_k^{(2)})^* y] = 0;$

for all $y \in L$, $x_i, x_j, x_k, x_r \in \mathfrak{B}$ and $i \leq j \leq k \leq r$.

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A Lie ring L is **4-Engel** if and only if

(1) $[(x_i^{(4)})^* y] = 0;$

(2) $[(x_i^{(3)} x_j)^* y] + [(x_i^{(2)} x_j^{(2)})^* y] + [(x_i x_j^{(3)})^* y] = 0;$

(3) $[(x_i^{(2)} x_j x_k)^* y] + [(x_i x_j^{(2)} x_k)^* y] + [(x_i x_j x_k^{(2)})^* y] = 0;$

(4) $[(x_i x_j x_k x_r)^* y] = 0;$

(5) $2[(x_i^{(2)} x_j^{(2)})^* y] = 0;$

(6) $2[(x_i x_j^{(2)} x_k)^* y] = 0;$

(7) $2[(x_i x_j x_k^{(2)})^* y] = 0;$

for all $y \in L$, $x_i, x_j, x_k, x_r \in \mathfrak{B}$ and $i \leq j \leq k \leq r$.

(4) with $i \leq j = k \leq r$ implies (6).

4-Engel Lie rings

A Lie ring L is **4-Engel** if and only if

(1) $[(x_i^{(4)})^* y] = 0;$

(2) $[(x_i^{(3)} x_j)^* y] + [(x_i^{(2)} x_j^{(2)})^* y] + [(x_i x_j^{(3)})^* y] = 0;$

(3) $[(x_i^{(2)} x_j x_k)^* y] + [(x_i x_j^{(2)} x_k)^* y] + [(x_i x_j x_k^{(2)})^* y] = 0;$

(4) $[(x_i x_j x_k x_r)^* y] = 0;$

(5) $2[(x_i^{(2)} x_j^{(2)})^* y] = 0;$

(6) $2[(x_i x_j^{(2)} x_k)^* y] = 0;$

(7) $2[(x_i x_j x_k^{(2)})^* y] = 0;$

for all $y \in L$, $x_i, x_j, x_k, x_r \in \mathfrak{B}$ and $i \leq j \leq k \leq r$.

(4) with $1 \leq j \leq k = r$ implies (7).

4-Engel Lie rings

A Lie ring L is **4-Engel** if and only if

(1) $[(x_i^{(4)})^* y] = 0;$

(2) $[(x_i^{(3)} x_j)^* y] + [(x_i^{(2)} x_j^{(2)})^* y] + [(x_i x_j^{(3)})^* y] = 0;$

(3) $[(x_i^{(2)} x_j x_k)^* y] + [(x_i x_j^{(2)} x_k)^* y] + [(x_i x_j x_k^{(2)})^* y] = 0;$

(4) $[(x_i x_j x_k x_r)^* y] = 0;$

(5) $2[(x_i^{(2)} x_j^{(2)})^* y] = 0;$

(6) $2[(x_i x_j^{(2)} x_k)^* y] = 0;$

(7) $2[(x_i x_j x_k^{(2)})^* y] = 0;$

for all $y \in L$, $x_i, x_j, x_k, x_r \in \mathfrak{B}$ and $i \leq j \leq k \leq r$.

(3) with $i \leq j = k$ is $2[(x_i^{(2)} x_j^{(2)})^* y] + 6[(x_i x_j^{(3)})^* y] = 0.$

4-Engel Lie rings

A Lie ring L is **4-Engel** if and only if

(1) $[(x_i^{(4)})^* y] = 0;$

(2) $[(x_i^{(3)} x_j)^* y] + [(x_i^{(2)} x_j^{(2)})^* y] + [(x_i x_j^{(3)})^* y] = 0;$

(3) $[(x_i^{(2)} x_j x_k)^* y] + [(x_i x_j^{(2)} x_k)^* y] + [(x_i x_j x_k^{(2)})^* y] = 0;$

(4) $[(x_i x_j x_k x_r)^* y] = 0;$

(5) $2[(x_i^{(2)} x_j^{(2)})^* y] = 0;$

(6) $2[(x_i x_j^{(2)} x_k)^* y] = 0;$

(7) $2[(x_i x_j x_k^{(2)})^* y] = 0;$

for all $y \in L$, $x_i, x_j, x_k, x_r \in \mathfrak{B}$ and $i \leq j \leq k \leq r$.

(3) with $i \leq j = k$ is $2[(x_i^{(2)} x_j^{(2)})^* y] + 6[(x_i x_j^{(3)})^* y] = 0.$

(4) with $1 \leq j = k = r$ is $6[(x_i x_j^{(3)})^* y] = 0.$

4-Engel Lie rings

A Lie ring L is **4-Engel** if and only if

(1) $[(x_i^{(4)})^* y] = 0$;

(2) $[(x_i^{(3)} x_j)^* y] + [(x_i^{(2)} x_j^{(2)})^* y] + [(x_i x_j^{(3)})^* y] = 0$;

(3) $[(x_i^{(2)} x_j x_k)^* y] + [(x_i x_j^{(2)} x_k)^* y] + [(x_i x_j x_k^{(2)})^* y] = 0$;

(4) $[(x_i x_j x_k x_r)^* y] = 0$;

(5) $2[(x_i^{(2)} x_j^{(2)})^* y] = 0$;

(6) $2[(x_i x_j^{(2)} x_k)^* y] = 0$;

(7) $2[(x_i x_j x_k^{(2)})^* y] = 0$;

for all $y \in L$, $x_i, x_j, x_k, x_r \in \mathfrak{B}$ and $i \leq j \leq k \leq r$.

(3) with $i \leq j = k$ is $2[(x_i^{(2)} x_j^{(2)})^* y] + 6[(x_i x_j^{(3)})^* y] = 0$.

(4) with $1 \leq j = k = r$ is $6[(x_i x_j^{(3)})^* y] = 0$.

Subtracting we obtain (5).

4-Engel Lie rings

A Lie ring L is **4-Engel** if and only if

- $[(x_i^{(4)})^* y] = 0;$
- $[(x_i^{(3)} x_j)^* y] + [(x_i^{(2)} x_j^{(2)})^* y] + [(x_i x_j^{(3)})^* y] = 0;$
- $[(x_i^{(2)} x_j x_k)^* y] + [(x_i x_j^{(2)} x_k)^* y] + [(x_i x_j x_k^{(2)})^* y] = 0;$
- $[(x_i x_j x_k x_r)^* y] = 0;$

for all $y \in L$, $x_i, x_j, x_k, x_r \in \mathfrak{B}$ and $i \leq j \leq k \leq r$.

n-Engel Lie rings

A Lie ring L is ***n*-Engel** if and only if

$$\sum_{\substack{k_1,\ldots,k_s\geq 1\\ k_1+\ldots+k_s=n}} [(x_{j_1}^{(k_1)}\cdots x_{j_s}^{(k_s)})^* y] = 0$$

for all $y \in L$, $1 \leq s \leq n$, $1 \leq j_1 \leq \ldots \leq j_s \leq m$.

Example

A Lie ring L is **4-Engel** if and only if

- $[(x_i^{(4)})^* y] = 0;$
- $[(x_i^{(3)} x_j)^* y] + [(x_i^{(2)} x_j^{(2)})^* y] + [(x_i x_j^{(3)})^* y] = 0;$
- $[(x_i^{(2)} x_j x_k)^* y] + [(x_i x_j^{(2)} x_k)^* y] + [(x_i x_j x_k^{(2)})^* y] = 0;$
- $[(x_i x_j x_k x_r)^* y] = 0;$
- $2[(x_i^{(2)} x_j^{(2)})^* y] = 0;$
- $6[(x_i x_j^{(3)})^* y] = 0;$
- $2[(x_i x_j^{(2)} x_k)^* y] = 0;$
- $2[(x_i x_j x_k^{(2)})^* y] = 0;$

for all $y \in L$, $x_i, x_j, x_k, x_r \in \mathfrak{B}$ and $i < j < k < r$.

Implementation

We have implemented the algorithms in the computer algebra systems **GAP4** and **Magma**.

Implementation

We have implemented the algorithms in the computer algebra systems **GAP4** and **Magma**.
Using this implementation we have obtained the lower central series of 3-Engel Lie rings with 2, 3 and 4 generators, 4-Engel Lie rings with 2 generators and 5-Engel Lie rings with 2 generators.

Lower central series dimensions

3-Engel	2 *gens*	3 *gens*	4 *gens*
L^1	8	60	541
L^2	6	57	537
L^3	5	54	531
L^4	3	46	511
L^5	2	36	472
L^6	-	18	388
L^7	-	9	293
L^8	-	3	173
L^9	-	-	62
L^{10}	-	-	18
L^{11}	-	-	4

Table: Lower central series dimensions

Lower central series dimensions

4-Engel	2 *gens*
L^1	34
L^2	32
L^3	31
L^4	29
L^5	26
L^6	24
L^7	20
L^8	16
L^9	12
L^{10}	6
L^{11}	3
L^{12}	1

5-Engel	2 *gens*
L^1	72
L^2	70
L^3	69
L^4	67
L^5	64
L^6	58
L^7	52
L^8	40
L^9	32
L^{10}	24
L^{11}	12
L^{12}	6
L^{13}	2

Table: Lower central series dimensions

Thank you!