Lie methods in Engel groups

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- But in 5-Engel groups the normal closure of an element need not be nilpotent

A construction

Let B be the free group on $b_1, b_2, \ldots$ in the variety of groups which are nilpotent of class 2 and have exponent 3.
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Let N be the normal closure of A in G.

Then N is an elementary abelian 3-group with basis $\{a^b \mid b \in B\}$, and

$$[a, [b_1, b_2], [b_1, b_3], \ldots, [b_1, b_k]] \neq 1$$

for any k. So the normal closure of b_1 in G is not nilpotent.

G is 5-Engel

Let $w, x, y, z \in G$. Then $[w, x, y]$ and $z^3 \in N$, and so

$$\begin{aligned} 1 &= [[w, x, y], z^3] \\ &= [w, x, y, z]^3 [w, x, y, z, z]^3 [w, x, y, z, z, z] \\ &= [w, x, y, z, z, z] \end{aligned}$$

Associated Lie rings

If G is a group we define the associated Lie ring of G as follows. We form the lower central series

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \gamma_3(G) \geq \dots,$$

where $\gamma_{i+1}(G)$ is the subgroup of G generated by $\{[g, h] \mid g \in \gamma_i(G),\ h \in G\}$.

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For $i = 1, 2, \dots$ we let $L_i = \gamma_i(G)/\gamma_{i+1}(G)$. Then L_i is an abelian group, and we think of it as a $\mathbb{Z}$-module and write

$$L = L_1 \oplus L_2 \oplus L_3 \oplus \dots,$$

a direct sum of $\mathbb{Z}$-modules.

Associated Lie rings

If $a = g\gamma_{i+1}(G) \in L_i$ and $b = h\gamma_{j+1}(G) \in L_j$ then we set

$$[a, b] = [g, h]\gamma_{i+j+1}(G) \in L_{i+j}$$

We extend this Lie product to the whole of L by linearity

Associated Lie rings

Problem

Is the associated Lie ring of an n-Engel group necessarily an n-Engel Lie ring?

Associated Lie rings

If L is the associated Lie ring of an n-Engel group then

- If $a \in L_i$ and $b \in L_j$, then $[a, \underbrace{b, b, \ldots, b}_{n}] = 0$
- If $a, b_1, b_2, \ldots, b_n \in L$ then

$$\sum_{\sigma \in \mathrm{Sym}(n)} [a, b_{1\sigma}, b_{2\sigma}, \ldots, b_{n\sigma}] = 0$$

Associated Lie rings

So Zel'manov's Theorem implies that if G is an n-Engel group, and if L is the associated Lie ring of G, then L and $G/\bigcap_{i\geq 1}\gamma_i(G)$ are locally nilpotent.

The first relation above follows directly from the definition of L

To see the second relation, we substitute $y_1y_2\ldots y_n$ for y in the group relation $[x,\underbrace{y,y,\ldots,y}_{n}]=1$. Expanding, we have

$$\prod_{\sigma\in\mathrm{Sym}(n)}[x,y_{1\sigma},y_{2\sigma},\ldots,y_{n\sigma}]\in\gamma_{n+2}(G)$$

The second relation follows immediately

Another useful relation

Substitute x_1x_2 for x in the group relation, and we have

$$\begin{aligned} 1 &= [x_1x_2, y, y, \ldots, y] \\ &= [[x_1, y][x_1, y, x_2][x_2, y], y, \ldots, y] \\ &= [x_1, y, y, \ldots, y][x_1, y, x_2, y, \ldots, y][x_2, y, y, \ldots, y] \end{aligned}$$

modulo $\gamma_{n+3}(G)$

So

$$[x_1, y, x_2, y, \ldots, y] \in \gamma_{n+3}(G)$$

This implies that L satisfies

$$\sum_{\sigma \in \mathrm{Sym}(n)} [a, b_{1\sigma}, c, b_{2\sigma}, \ldots, b_{n\sigma}] = 0$$

Even more relations

If we substitute $x_1 x_2 \ldots x_r$ for x in the group relation, and substitute $y_1 y_2 \ldots y_s$ for y (with $s \geq n$), then we obtain a multilinear Lie relation of weight $r + s$.

So when we move to Lie rings the n-Engel group identity gives Lie relations which are both stronger and weaker than the n-Engel Lie identity.

Locally nilpotent n-Engel groups

We use Lie methods to study finitely generated nilpotent n-Engel groups.

Since finitely generated nilpotent groups are residually finite p-groups, we study them by studying finite n-Engel p-groups.

If G is a finite p-group with associated Lie ring L, then L has the same order, class, and number of generators as G. Furthermore L/pL has the same class of L and can be thought of as a Lie algebra over $\mathbb{Z}_p$.

Global nilpotence

In characteristic zero the Lie identity

$$\sum_{\sigma \in \mathrm{Sym}(n)} [a, b_{1\sigma}, b_{2\sigma}, \ldots, b_{n\sigma}] = 0$$

implies *global nilpotence*. (Zel'manov again!)

So, for some c, in the free Lie algebra F over the rationals $\mathbb{Q}$, with free generators $x_1, x_2, \ldots, x_{c+1}$ we have

$$[x_1, x_2, \ldots, x_{c+1}] = \sum_{i=1}^{k} \alpha_i \left(\sum_{\sigma \in \mathrm{Sym}(n)} [a_i, b_{i1\sigma}, b_{i2\sigma}, \ldots, b_{in\sigma}] \right)$$

for some rationals α_i, and some elements $a_i, b_{ij} \in F$

Global nilpotence

If p does not divide the denominators of any of the α_i, this implies that a finite n-Engel p-group is nilpotent of class at most c.

So, if $n \geq 2$ there are integers P and c depending on n, such that if G is a finite n-Engel p-group for any $p > P$, then G is nilpotent of class at most c.

For small n the numbers P and c are surprisingly small.

n	P	c
2	1	3
3	5	4
4	5	7
5	7?	10?

Higman's solution of RBP for exponent 5

If G is a finite m-generator group of exponent 5 with associated Lie ring L then

- L is m-generator
- $|G| = |L|$
- G and L have the same nilpotency class
- L satisfies the identical relations

$$5x = 0$$

$$\sum_{\sigma \in \mathrm{Sym}(4)} [x, y_{1\sigma}, y_{2\sigma}, y_{3\sigma}, y_{4\sigma}] = 0$$

Higman's solution

To solve RBP for exponent 5 it is sufficient to show that Lie algebras over $\mathbb{Z}_5$ satisfying

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In fact the bound is 6

Higman's reduction

Let F be the free 4-Engel Lie algebra over $\mathbb{Z}_5$ with free generators $x, a_1, a_2, a_3, \ldots$.

Let I be the ideal of F generated by $\{[a_i, a_j] \mid i, j \geq 1\}$.

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This, in turn, implies that if G is a finite m-generator group of exponent 5, then G is nilpotent of class at most $6m$.

Application of Higman's reduction

Applying Higman's reduction to finite n-Engel p-groups, we see that the nilpotency class of the normal closure of an element has the following bounds.

	$p=2$	$p=3$	$p=5$	$p=7$	$p>7$
$n=2$	1	1	1	1	1
$n=3$	2	2	2	2	2
$n=4$	4	3	4	3	3
$n=5$	?	∞	6	5	4?