The structure of thin Lie algebras up to the second diamond

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Thin Lie algebras

A graded Lie algebra

$$L = \bigoplus_{i=1}^{\infty} L_i$$

over a field $\mathbb{F}$ is said to be thin if $\dim(L_1) = 2$ and the following *covering property* holds:

$$[uL_1] = L_{i+1} \quad \text{for all } 0 \neq u \in L_i, \text{ for all } i \geq 1$$

The covering property can be reformulated as follows:

for every graded ideal I in L, $L^{i+1} \subseteq I \subseteq L^i$, for some i.

[A. Caranti, S.Mattarei, M.Newman, C.Scoppola, *Thin groups of prime power order and thin Lie algebras*, 1996]

A p-group G is said to be thin if $\gamma_i(G)/\gamma_{i+1}(G)$ is elementary abelian of order p or p^2 and for every nontrivial normal subgroup N of G

$$\gamma_{i+1}(G) \leq N \leq \gamma_i(G)$$

for some i.

[R. Brandl, *The Dilworth number of subgroup lattices*, 1988]

The definition of thin group extends to pro p-groups

[A. Caranti, S. Mattarei, M. Newmann, C. Scoppola, *Thin groups of prime power order and Lie algebras*,1996]

Examples of thin groups:

* groups of maximal class

* the so-called *Nottingham group* over the field $\mathbb{F}_p$

The Nottingham group

$$J = \{t + \sum_{i=2}^{\infty} a_i t^i : a_i \in \mathbb{F}_p\}$$

where for $\alpha = t + \sum_{i=2}^{\infty} a_i t^i$ and $\beta \in J$

$$\alpha * \beta = \beta + \sum_{i=2}^{\infty} a_i \beta^i$$

G is thin if and only if

$$L(G) = \bigoplus_{i=1}^{\infty} \gamma_i(G)/\gamma_{i+1}(G)$$

is a thin Lie algebra.

[A. Caranti, S. Mattarei, M. Newmann, C. Scoppola, *Thin groups of prime power order and thin Lie algebras*,1996]

Thin groups are group of *width* two and *obliquity zero*

[G. Klass, C. R. Leedham - Green, W. Plesken, *Linear pro p-groups of finite width*, 1997]

Let L be a thin Lie algebra.

L is generated (as Lie algebra) by L_1

L is centerless if infinite dimensional and the center of L is the highest nonzero homogeneous component otherwise

Every homogeneous component of L has dimension 1 or 2 (or possibly zero if L is finite dimensional)

An homogeneous component of dimension two is termed a *diamond* of L.

In particular L_1 is a diamond of L

Thin Lie algebras with no diamonds except for L_1 are *(graded) Lie algebras of maximal class (generated in degree* 1*)*

(Graded) Lie algebras of maximal class

$$L = \bigoplus_{i=1}^{\infty} L_i$$

is a Lie algebra of maximal class if $\dim(L_1) = 2$, $\dim(L_i) \leq 1$ and $[L_i, L_1] = L_{i+1}$.

$*$ $L^i = \bigoplus_{j \geq i} L_j$ thus (if $L_{i-1} \neq 0$) $\dim(L/L^i) = i$ and $c(L/L^i) = i - 1$.

In characteristic zero there is a unique infinite dimensional Lie algebra of maximal class and it is metabelian.

In prime characteristic there are uncountably many isomorphism classes of infinite dimensional Lie algebras of maximal class

Classification.

[A. Caranti, M. Newman, *Graded Lie algebras of maximal class II*, 2000] p odd

[G. Jurman, *Graded Lie algebras of maximal class III*, 2005] $p = 2$

The *two-step* centralizers of L:

$$C_i = C_{L_1}(L_i) = \{a \in L_1 \ : \ [a, b] = 0, \text{for all } b \in L_i\}$$

thus $\dim(C_i) = 2 - \dim(L_{i+1})$.

The sequence $\{C_i = C_{L_1}(L_i)\}_{i \geq 2}$ determines L up to isomorphism.

Choose $y \in L_1$ such that $\mathbb{F}y = C_2$:

$*$ if $C_i = \mathbb{F}y$ for every $i \geq 2$ (or possibly $C_i = L_1$ when L is f.d.) then L is metabelian;
$*$ otherwise let $\mathbb{F}x = C_{\overline{q}}$ be the second distinct centralizer in order of occurrence, then $\overline{q} = 2p^s$ ($p > 0$ the characteristic of the underlying field)

$*$ the sequence $\{C_i\}_{i\geq 2}$ consists of consecutive occurrences of $\mathbb{F}y$ interrupted by isolated occurrences of other centralizers.

Suppose that

$C_i \neq \mathbb{F}y,\ C_{i+1} = \ldots = C_{i+m-1} = \mathbb{F}y,\ C_{i+m} \notin \{\mathbb{F}y, L_1\}$
then $\{C_{i+1}, \ldots, C_{i+m}\}$ is a *constituent* of L of length m

The first constituent $\{C_2, \ldots C_{\overline{q}}\}$ is conventionally said to be of length $\overline{q}$.

The possible values for m are

$$2p^s \quad \text{or} \quad 2p^s - p^h,\ \ 0 \leq h \leq s$$

provided L is infinite dimensional or of dimension large enough.

In the special case L has exactly two distinct two-step centralizers then L is uniquely determined by the sequence of its constituent lenghts.

[A. Caranti, S.Mattarei, M. Newman, *Graded Lie algebras of maximal class*, 1997]

Second diamond of thin algebras

Let L be a thin Lie algebra and suppose that L is not of maximal class. Let L_k be the second diamond of L.

Suppose that $L = L(G)$ for an infinite thin pro p-group G, then $L/L^k = L_1 \oplus \cdots \oplus L_{k-1}$ is a metabelian Lie algebra of maximal class, p is odd and k can only be 3, 5 or p.

[A. Caranti, S. Mattarei, M. Newman, C. Scoppola, *Thin groups of prime power order and thin Lie algebras*,1996]

Pro p-groups such that the corresponding Lie algebra has second diamond in degree $k = 3$ or 5 have been studied in

[S. Mattarei, *Some thin pro p-groups*, 1999]

The case $k = p$ occurs for the graded Lie algebra associated to the Nottingham group (w.r.t. its lower central series)

Thin Lie algebras not necessarily coming from thin groups.

Theorem [A. Caranti and G. Jurman]

Let L be a thin Lie algebra over a field of odd characteristic, with second diamond L_k and dimension large enough. Then L/L^k is metabelian.

[A. Caranti, G. Jurman, *Quotients of maximal class of thin Lie algebras*, 1999]

Theorem

Let L be a thin Lie algebra over a field of odd characteristic, with second diamond L_k and dimension large enough. Then the possible values for k are 3,5, p^s or $2p^s - 1$.

[A. Caranti,S. Mattarei, *Some Thin Lie algebras related to Albert-Frank algebras and algebras of maximal class*, 1999]

[M. A, G. Jurman, *Diamonds in thin Lie algebras*, 2001]

Theorem

Let L be a thin Lie algebra, over a field of characteristic two, with second diamond L_k and dimension larger than $(4k+1)/3$. Suppose that the quotient L/L^k is not metabelian, but L/L^6 is. Let $\mathbb{F}y = C_{L_1}(L_2)$ and $\mathbb{F}x = C_{L_1}(L_{\overline{q}})$ be the first two distinct two-step centralizers, at their first occurrence and set $M = L/([L_{k-1}x] + L^{k+1})$. Then

1. the graded Lie algebra of maximal class M has exactly two distinct two-step centralizers;

2. the sequence of constituent lengths of M is either $\overline{q}, \overline{q}-2$, or $\overline{q}, \overline{q}-1, \overline{q}\ ^{2r-3}, \overline{q}-1$, where $\overline{q}$ and r are powers of two;

3. $k+1$ is a power of two.

All the cases in the Theorem do really occur.

Corollary

The second diamond of a thin Lie algebra L in characteristic p, with L/L^6 metabelian and the dimension of L large enough, allways occurs in odd degree of the form p^n or $2p^n - 1$ for some $n > 0$.

Partial results in

[G. Jurman, *Quotients of maximal class of thin Lie algebras. The characteristic two case*, 1999]