RESTRICTED LIE ALGEBRAS IN WHICH EVERY RESTRICTED SUBALGEBRA IS AN IDEAL

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R. Dedekind: *Über Gruppen, deren sämmtliche Theiler Normaltheiler sind*, Mathematische Annalen **48** (1897), 548-561.

R. Baer: *Situation der Untergruppen und Struktur der Gruppe*, S. B. Heidelberg. Akad. Wiss. **2** (1933), 12-17.

A *Dedekind group* is a group in which all subgroups are normal.

A non–abelian Dedekind group is called *Hamiltonian*.

Theorem

A group G is Hamiltonian if and only if

$$G = Q_8 \times E \times D,$$

where Q_8 is a quaternion group of order 8, E an elementary abelian 2–group, and D an abelian group with all its elements of odd order.

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Proposition

Let $(L, [p])$ be a non–abelian restricted Lie algebra over a field F of characteristic $p > 0$ such that every restricted subalgebra of L is an ideal. Then one has:

1. $L' \subseteq L^{[p]} \subseteq Z(L)$;
2. $\mathcal{N}(L) \subseteq Z(L)$;
3. every element of L is p-algebraic.

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Theorem (S., Proc. Amer. Math. Soc., to appear)

Let $(L, [p])$ be a restricted Lie algebra over a perfect field F of characteristic $p > 0$. Then every restricted subalgebra of L is an ideal if and only if one of the following conditions is satisfied:

(i) L is abelian;

(ii) $p = 2$, L is nilpotent of class 2 and $L = T \oplus H$, where T is a torus and H a 2-nil restricted Lie algebra such that H_2' is cyclic, $H_2' = H^{[2]}$, and $\mathcal{N}(H) \subseteq Z(H)$.

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Sketch of the proof.

Sufficiency.

Assume that condition (ii) of the statement holds and let J be a restricted subalgebra of L.
Let $x \in J$ and write

$$x = x_s + x_n$$

with $x_s \in T$ and $x_n \in H$. Then $x_n \in \langle x \rangle_2 \subseteq J$.

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Now, let h be any non–central element of H and suppose, if possible, that $h^{[2]} \in H^{[2]^2}$. We have $h^{[2]} = z^{[2]}$ for some $z \in Z(H)$, so that

$$h + z \in \mathcal{N}(H) \subseteq Z(H),$$

a contradiction.
It follows that $H_2' = \langle h^{[2]} \rangle_2$, thus

$$[x, L] = [x_n, H] \subseteq \langle x_n^{[2]} \rangle_2 \subseteq J.$$

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a contradiction.
It follows that $H'_2 = \langle h^{[2]} \rangle_2$, thus

$$[x, L] = [x_n, H] \subseteq \langle x_n^{[2]} \rangle_2 \subseteq J.$$

Necessity.

The odd characteristic case.

For every $a \in L$ consider the Jordan–Chevalley decomposition

$$a = a_s + a_n.$$

Let x be a p-nilpotent element of L of exponent n, say.
Suppose that there is a p-nilpotent element y of L such that $[x, y] \neq 0$.
We can assume that y has exponent $m \geq n$.
Since $[x, y] \in \langle x \rangle_p$, we have

$$[x, y] = \sum_{i=0}^{n-1} k_i x^{[p]^i}$$

for suitable $k_i \in F$. Let $r = \min\{i | \, k_i \neq 0\}$.

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Then we have

$$[x, y] = \left(\sum_{i=r}^{n-1} \alpha_i x^{[p]^{i-r}} \right)^{[p]^r}$$

for suitable $\alpha_i \in F$. Furthermore

$$[x, y] = \left(\sum_{j=m-n+r}^{m-1} \beta_j y^{[p]^{j-r}} \right)^{[p]^r}$$

for suitable $\beta_{m-n+r}, \ldots, \beta_{m-1} \in F$.
The element

$$g := \sum_{j=0}^{n-r-1} \left(\alpha_{j+r} x^{[p]^j} - \beta_{j+m-n+r} y^{[p]^{j+m-n}} \right)$$

satisfies $g^{[p]^r} = 0$.
Thus

$$0 = [g, y] = \alpha_r [x, y],$$

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$$H := \{x \in L | \, x \text{ 2-nilpotent}\}$$

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Then:

- T and H are ideals of L;
- $L = T \oplus H$;
- $\mathcal{N}(H) \subseteq Z(H)$;
- elements of H having different exponents commute;
- for every non–commuting elements x and y of H one has $\langle [x, y] \rangle_2 = \langle x^{[2]} \rangle_2 = \langle y^{[2]} \rangle_2$;
- $H_2' = \langle \bar{x}^{[2]} \rangle_2$ for any non–central element $\bar{x}$ of H;
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A field F of characteristic $p > 0$ is said to be *p-closed* if F has no extension of degree p.

Theorem

Let F be a perfect field of characteristic 2. Then the following conditions are equivalent:

1. there exists a non–abelian restricted Lie algebra over F with the property that all its restricted subalgebras are ideals;
2. F is not 2-closed.

Corollary

Let $(L, [p])$ be a restricted Lie algebra over an algebraically closed field of characteristic $p > 0$.Then every restricted subalgebra of L is an ideal if and only if L is abelian.

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Let $(L, [p])$ be a restricted Lie algebra over an algebraically closed field of characteristic $p > 0$. Then every restricted subalgebra of L is an ideal if and only if L is abelian.

A field F of characteristic $p > 0$ is said to be *p-closed* if F has no extension of degree p.

Theorem

Let F be a perfect field of characteristic 2. Then the following conditions are equivalent:

1. there exists a non–abelian restricted Lie algebra over F with the property that all its restricted subalgebras are ideals;
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Example

Let F be a field of characteristic $p > 2$ containing an element α with no p-th root in F.

Consider the restricted Lie algebra L over F with a basis $\{x, y, z\}$ such that $[x, y] = z$, $[x, z] = [y, z] = 0$, $x^{[p]} = z$, $y^{[p]} = \alpha z$, and $z^{[p]} = 0$.

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