Induced modules for modular Lie algebras

embedding into wreath products

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Outline

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1. Group actions and representations
   - Imprimitive G-sets
   - Sylow p-subgroups of $\mathrm{Sym}(p^h)$
   - Induced modules

Imprimitive actions
Blocks

- G finite group, X finite transitive G-set,
- Let $B \subset X$ be an imprimitivity block, $S = \{g \in G \mid Bg = B\}$ the stabilizer of B and T a set of right cosets representatives of S in G.

One has

- if $t, t' \in T$ with $t \neq t'$ then $B \cap Bt = \emptyset$
- $X = \dot{\bigcup}_{t \in T} Bt$
- $X \cong B \times T$ as sets
- $T \cong G/S$ as G-set where the action of $g \in G$ over $t \in T$ is denoted by t_g. The kernel of this action is the intersection $N = \cap_{g \in G} S^g$ of the conjugate subgroups of S (core of S).

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The action

- Let $b \in B$, $t \in T$ and $g \in G$ the action of g on $x = bt \in X$ is given by
$$xg = (bt)g = (bs_g)t_g \text{ where } s_g = tgt_g{}^{-1} \in S$$
- There is a class of groups acting transitively on $X \cong B \times T$ having the same imprimitivity block system

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Wreath products

- Let $L \leq \mathrm{Sym}(T)$ and $M \leq \mathrm{Sym}(B)$ be transitive permutation groups
- The set of functions $M^T = \{f \mid f : T \to M\}$ is a group under pointwise multiplication and L acts on it via the coinduced action $f^l(t) = f(tl^{-1})$
- The wreath product $M \wr_T L = L \ltimes M^T$ acts on $X \cong B \times T$ via
$$(b, t) \cdot (l, f) = (bf(tl), tl) = (bf^{l^{-1}}(t), tl)$$
and the fibers $B \times \{t\}$ form an imprimitivity block system for this action
- In particular the group $S \wr_T (G/N)$ acts on X.

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Kaloujnin Krasner embedding

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- The map $kk\colon G \to S \wr (G/N)$ is an embedding of permutation groups preserving the imprimitivity block system. This map is also known as *Kaloujnin Krasner embedding* (1951)
- When S is a normal subgroup one has that $N = S$ so that this map embeds G in $S \wr (G/S)$.

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   - Induced modules

Iterated kk-embedding

- If G is a p-group and $|X| = p^h$ then it is well known that there is a chain of imprimitivity blocks

$$\{x\} \subset B_1 \subset \cdots \subset B_{h-2} \subset X$$

such that $|B_i| = p^i$

- Iterating the kk embedding procedure along this chains one gets an embedding of G as a permutation subgroup of $W_h := \underbrace{C_p \wr \cdots \wr C_p}_{h \text{ times}}$
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Induced modules

- Let V be an S-module and $U = 1\uparrow_S^G = \langle T\rangle$ the permutation G-module with stabilizer S.
- $\bar{V} := V \otimes_K U$ is an $S \wr (G/N)$-module via the action

$$(v \otimes t) \cdot (Ng, f) = (vf(t_g), t_g)$$

- The restriction of $\bar{V}$ from $S \wr (G/N)$ to G (via the kk embedding) is the induced module $V\uparrow_S^G$.

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Sylow p-subgroup of $\mathrm{SL}((p-1)p^{s-1}, \mathbb{Z})$

Vol'vačev's result.

Theorem (Vol'vačev, 1967)

The iterated wreath product

$$W_s = \underbrace{C_p \wr \cdots \wr C_p}_{s \text{ times}}$$

is a p-Sylow subgroup of $\mathrm{SL}((p-1)p^{s-1}, \mathbb{Z})$

How Do We Find The Embedding $G/F \leq W_s$?

Main igredients of Vol'vačev's result.

Sketch of Vol'vačev's result

- If G is a finite p-group every $\mathbb{Q}(\xi)$-irreducible representation (ξ a p-th root of 1) is monomial, i.e. it is induced by a linear one λ of a subgroup H.
- Find a composition setries $S = G_0 \leq \cdots \leq G_{s-1} = G$ and use iteratively the kk map to get an embedding of G in the group $H \wr \underbrace{C_p \wr \cdots \wr C_p}_{s-1 \text{ times}}$ acting (unfaithfully) over $\mathbb{Q}(\xi)^{p^{s-1}}$.
- Possibly factoring out the kernel the representation we get an embedding of G in W_s acting on $\mathbb{Q}^{(p-1)p^{s-1}} \cong \mathbb{Q}(\xi)^{p^{s-1}}$ and the representation afforded is actually a $\mathbb{Z}$-representation.

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The Lie ring of $G \wr C_p$

$L(G \wr C_p)$

C. Di Pietro noticed in her thesis (2005) that if the Lie ring of G is a Lie algebra over $\mathbb{F}_p$ then

$$L(G \wr C_p) = (L(G) \otimes_{\mathbb{F}_p} A(1,1)) \rtimes \langle 1 \otimes D \rangle$$

where

$$A(1,1) = \mathbb{F}_p(\epsilon^{(1)}, \ldots, \epsilon^{(p-1)})$$

is the divided power algebra of height 1:

$$\epsilon^{(i)}\epsilon^{(j)} = \binom{i+j}{j}\epsilon^{(i+j)}$$

$$\epsilon^{(0)} := 1$$

and

$$D(\epsilon^{(i)}) = \epsilon^{(i-1)}$$

The Wreath Product $L \wr K$

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It is then natural to define for a Lie algebra L over the field K (of characteristic p)

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Note that if V is an L-module then

$$\mathrm{Wr}(V) := V \otimes_K A(1,1)$$

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A kk like embedding

Di Pietro's work

Theorem (Di Pietro 2005)

Let L be a restricted Lie algebra and I a p-ideal of codimension 1 containing $C(L)$. Suppose further that V is an I-module such that the induced representation $V \uparrow_I^L$ is faithful and has character S and that there exists an element $u \in C(L)$ acting as the identity on V. There exists an embedding $i \colon L \to I \wr (L/I)$ of (unrestricted) Lie algebras (possibly depending on the character S) such that

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Proof.

Write $L =< x > \oplus I$. The embedding is defined by

$$\begin{cases} \quad x & \mapsto 1 \otimes D + (x^{[p]} + S(x)^p u) \otimes \epsilon^{(p-1)} \\ I \ni y & \mapsto \sum_{j=0}^{p-1} [y, \underbrace{x, \dots, x}_{j \text{ times}}] \otimes \epsilon^{(j)} \end{cases}$$

□

Irreducible linear Lie algebras
Absolute irreducibility

Definition

A representation $\rho\colon L \to \mathfrak{gl}(V)$ is said to be absolutely irreducible if any of the following equivalent conditions is true:

1. $\rho(L)$ generate $\mathfrak{gl}(V)$ as associative algebra with unity
2. $C_{\mathfrak{gl}(V)}(\rho(L)) = F \cdot 1$
3. $V \otimes_K E$ is an irreducible $(L \otimes_K E)$-module for every field extension E/K.

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A linear algebra $L \leq \mathfrak{gl}(V)$ is said to be irreducible if V is an absolutely irreducible L-module

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2. $C_{\mathfrak{gl}(V)}(\rho(L)) = F \cdot 1$
3. $V \otimes_K E$ is an irreducible $(L \otimes_K E)$-module for every field extension E/K.

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A linear algebra $L \leq \mathfrak{gl}(V)$ is said to be irreducible if V is an absolutely irreducible L-module

Irreducible linear Lie algebras

Absolute irreducibility

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An Example

$K \wr K$ as irreducible nilpotent Lie algebra

$K \wr K$

The Lie algebra $K \wr K$ is generated over K by the $p \times p$ matrices

$$x = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \text{ and } y = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

It has nilpotency class p and $\dim_K(L) = p + 1$, i.e. it is an algebra of maximal class. Moreover $e_{ij} = x^{p-i}yx^{j-1}$ so that the associative algebra generated by $K \wr K$ is $\mathfrak{gl}(p, K)$ i.e. $K \wr K$ is an irreducible Lie algebra. More generally $L \wr K$ is irreducible if L is, so that $\wr^n K$ is irreducible for all n.

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The kk embedding

Kaloujnine Krasner embedding

The induced modules techniques provide a general formula for a Kaloujnine Krasner embedding. A different approach can be found in

Petrogradsky, V. M.; Razmyslov, Yu. P.; Shishkin, E. O. *Wreath products and Kaluzhnin-Krasner embedding for Lie algebras.* Proc. Amer. Math. Soc. 135 (2007), no. 3, 625–636.

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Outline

Inducing Representations From An Ideal

Universal reduced p-enveloping algebras

Let L be a finite dimensional restricted Lie algebra, I a p-ideal of L. Suppose that V is a restricted I-module with character S and that $u(L, S)$ and $u(I, S|_I)$ are the corresponding reduced universal p-enveloping algebras. We shall assume further that $C(L)$ is a subalgebra of I containing an element u acting on V as the identity

Let $(\mathrm{e}_{k+1}, \ldots, \mathrm{e}_n)$ be a basis of I such that $(\mathrm{e}_1, \ldots, \mathrm{e}_k, \mathrm{e}_{k+1}, \ldots, \mathrm{e}_n)$ is a basis of L. If $\tau = (p-1, \ldots, p-1)$ and $\mathbf{e} = (\mathrm{e}_1, \ldots, \mathrm{e}_k)$, then (with multindex notation)

$$u(L, S) = \bigoplus_{\mathbf{a} \leq \tau} \mathbf{e}^{\mathbf{a}} u(I, S|_I)$$

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Inducing Representations From An Ideal

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Since I is an ideal, the map $\Lambda\colon u(L,S) \to u(I,S|_I)$ defined by $(j_{\mathbf{a}} \in u(I,S|_I))$

$$\Lambda\left(\sum_{\mathbf{a}\leq\tau} \mathrm{e}^{\mathbf{a}} j_{\mathbf{a}}\right) = j_\tau$$

is right and left $u(I,S|_I)$-linear and surjective, indeed it makes $u(I,S|_I) \leq u(L,S)$ into a Frobenius extension.
As shown in Strade's book Simple Lie algebras over Fields of positive Characteristic, the map

$$\phi\colon u(L,S) \otimes_{u(I,S|_I)} V \to \hom_{u(I,S|_I)}(u(L,S),V)$$

defined by $\phi(u\otimes v)(m) = \Lambda(um)v$ is an isomorphism of $u(L,S)$ modules, so that it makes no difference in using induced or coinduced modules.

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The coinduced module

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There are canonical K-isomorphisms

$$\begin{aligned} \hom_{u(I,S|_I)}(u(L,S),V) &\cong \hom_K(u(L/I),V) \\ \text{(}V\text{ finite dimensional)} &\cong \hom_K(u(L/I),K)\otimes_K V \end{aligned}$$

Derivation algebras

Well known: $\hom_K(u(L/I),K)\cong A(k,\mathbf{1})$ is a divided power algebra on k variables and L/I acts on it as a (special) derivation algebra.

Define the structure

Now we use the previous K-isomorphisms to transfer the coinduced action on $\hom_K(u(L/I),V)$.

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The wreath product

The wreath product

For A and B restricted Lie algebras define

$$A \wr B := B \ltimes \hom_K(u(B), A)$$

where $(b'f)(b) = f(bb')$, for $f \in \hom_K(u(B), A)$ and $b' \in B$.

Lie products in $\hom_K(u(B), A)$

For $f, g \in \hom_K(u(B), A)$ the lie product $[f, g]$ is the map defined as $[f, g](u) = \sum [f(u_1), g(u_2)]$, in Sweedler's notation (being $\sum u_1 \otimes u_2$ the comultiplication of u in the Hopf algebra $u(B)$).

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The action

Clearly $I \wr (L/I) = (L/I) \ltimes \hom_K(u(L/I), I)$ acts on $\hom_K(u(L/I), V)$.
Note that for $g \in \hom_K(u(L/I), I)$ and $f \in \hom_K(u(L/I), V)$ one has

$$(g \cdot f)(\mathbf{e}^a) = \sum_{\mathbf{b} \leq \mathbf{a}} \binom{\mathbf{a}}{\mathbf{b}} g(\mathbf{e}^{\mathbf{b}}) f(\mathbf{e}^{\mathbf{a}-\mathbf{b}})$$

The Formula

The embedding

Let

$$[l, \mathbf{e}; \mathbf{b}] := [[[\cdots [l, \underbrace{\mathrm{e}_1, \dots, \mathrm{e}_1}_{b_1 \text{ times}}], \dots] \underbrace{\mathrm{e}_k, \dots, \mathrm{e}_k}_{b_k \text{ times}}].$$

We can now define $i \colon L \to L \wr (L/I)$ by setting

$$\begin{aligned} i(l) &:= (\mathbf{e}^{\mathbf{a}} \mapsto [l, \mathbf{e}; \mathbf{a}], 0) \text{ for } l \in I \\ i(l) &:= (\mathbf{e}^{\mathbf{a}} \mapsto t_a(l), l + I) \text{ for } l \in L \setminus I \end{aligned}$$

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Remark

- This embedding can be described in terms of divided powers (extends Di Pietro's result): $\hom_K(u(L/I), V) \cong A(k,1) \otimes_K V$.
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Consequences

Vol'vačev statement for Irreducible nilpotent linear Lie algebras

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A linear (absolutely) irreducible nilpotent Lie algebra L over a perfect field K is a subalgebra of $\wr^n K$ for some n.

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Vol'vačev statement for Irreducible nilpotent linear Lie algebras

Sketch of the proof (close to Strade Farnsteiner book section. 5.8)

- WLOG we can assume $L \leq \mathfrak{gl}(V)$ restricted and non-abelian
- As L is nilpotent there is a non-central abelian ideal $A \trianglelefteq L$ containing $C(L)$
- Since for $a \in A$ we have $[l, a^{p^h}] = [l, \underbrace{a, \ldots, a}_{p^h \text{ times}}] = 0$ so that a^{p^h} is central (h large enough), hence both a^{p^h} and a have eigenvalues in K
- $\exists \lambda \colon A \to K$ s.t. $a - \lambda(a)$ is nilpotent $\forall a \in A$

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